LECTURE NOTES DAY 19
ECE EE640

Optimum Decision Boundaries
MAP, Neyman Pearson, Lagrange Multipliers 3-31-05
Lecture 19

Types of Decision Boundaries

MAP (Maximum a posteriori probability)
(also referred to as Maximum Likelihood)

Decide $H_0$ or $H_1$ based on

Bayes Rule

$$P(H_i | y) = P(H_i) \cdot P(y | H_i)$$

Performance measure

$$P_e = P(H_0) \cdot P(D_1 | H_0) + P(H_1) \cdot P(D_0 | H_1)$$

where $D_i = \text{Decide } H_i$
The MAP decision rule can also be put in the form of a maximum likelihood ratio (MLR)

\[
\frac{P(H_1) f(y | H_1)}{P(H_0) f(y | H_0)} \geq \frac{1}{P(H_0)}
\]

\[
\Rightarrow \frac{f(y | H_1)}{f(y | H_0)} \geq \frac{P(H_0)}{P(H_1)}
\]

Cost Functions

Let \( C_{ij} \) be the cost associated with \( P(D_i, H_j) \)
Average cost is

\[ \bar{c} = \sum_{i=0,1} \sum_{j=0,1} c_{ij} \Pr(H_i) \Pr(D_i|H_j) \]

\[ = C_{00} \Pr(D_0|H_0) \Pr(H_0) + C_{10} \Pr(D_1|H_0) \Pr(H_0) + C_{01} \Pr(D_0|H_1) \Pr(H_1) + C_{11} \Pr(D_1|H_1) \Pr(H_1) \]

\[ = C_{00} \Pr(H_0) \int_{R_0} f_{y|H_0}(y|H_0) \, dy \]

\[ + C_{10} \Pr(H_0) \int_{R_0} f_{y|H_0}(y|H_0) \, dy \]

\[ + C_{01} \Pr(H_1) \int_{R_1} f_{y|H_1}(y|H_1) \, dy \]

\[ + C_{11} \Pr(H_1) \int_{R_1} f_{y|H_1}(y|H_1) \, dy \]
Cost of making an incorrect decision may be higher than $C_{10} > C_{11}$ and $C_{01} > C_{11}$. 

Note:

$R_i R_0 = \phi$ 
$R_i R_0 = U$ 

$R_i R_0$ and $C_{11}$ are constants.

$C_{10} = R_i P(H_0) + C_{11} P(H_{11})$ 

$C = R_0 \phi f_{Y_{11}} R_0 - R_{11} P(H_{11})$ 

Integrands in negative when

$\int_{-\infty}^{\infty} f_Y (y) \, dy$
\[ P(H_0)(C_{10} - C_{00}) f_{y|H_0}(y|H_0) > P(H_1)(C_{01} - C_{11}) f_{y|H_1}(y|H_1) \]

Choose \( H_0 \)

\[
\frac{P(H_1)(C_{01} - C_{11}) f_{y|H_1}(y|H_1)}{P(H_0)(C_{10} - C_{00}) f_{y|H_0}(y|H_0)} \geq 1
\]

\[ L(y) = \frac{f_{y|H_1}(y|H_1)}{f_{y|H_0}(y|H_0)} \geq \frac{P(H_0)(C_{10} - C_{00})}{P(H_1)(C_{01} - C_{11})} \]
If $P(H_0)$ and $P(H_1)$ are not known then we have the "minmax" rule.

Minimize expected cost given $P(H_1)$ for which average cost is maximum.

Neyman-Pearson rule (N-P)
nor $P(H_0)$ or $P(H_1)$ is known.
no cost assignments,

Constraint $P(D_0 | H_0) < \gamma$

"false alarm",

Minimize $P(D_0 | H_1) = "miss"$
The results for "minmax" and $N-P$ is

\[ L(y) = \frac{\int y_{1|H_1}(y|H_1) \; dy_{1|H_1}}{\int y_{1|H_0}(y|H_0) \; dy_{1|H_0}} \geq \chi \]

When ever you have a constraint and a minimization, think "Lagrange multipliers".

We can use Lagrange multipliers as a function

\[ F = P_m + \lambda \left[ P_F - \chi \right] \]

where we want to minimize $P_m$ and constrain $P_F = \chi$.

\[ P_m = \int_{R_0} f_{y|H_0}(y|H_0) \; dy \]

\[ P_F = \int_{R_1} f_{y|H_0}(y|H_0) \; dy \]
\[ F = \lambda (1 - \alpha') + \sum_{R_0} f_{y_{1:H_1}}(y_{1:H_1}) - \lambda \int f_{y_{1:H_0}}(y_{1:H_0}) \, dy \]

\[ \lambda > 0 \text{ will minimize} \]
\[ \lambda < 0 \text{ will minimize} \]

Let \( \frac{dF}{d\lambda} = 0 \) where \( \lambda = \frac{\gamma}{R_0} - \infty \)

\[ 0 = \frac{dF}{d\lambda} = 1 - \alpha' - \sum_{R_0} f_{y_{1:H_0}}(y_{1:H_0}) \, dy \]

Assuming \( f_{y_{1:H_1}}(y_{1:H_1}) = \lambda f_{y_{1:H_0}}(y_{1:H_0}) \)

\[ \Lambda(y) = \frac{f_{y_{1:H_1}}}{f_{y_{1:H_0}}} \frac{\Lambda_{H_1}}{\Lambda_{H_0}} \]

\[ L(y) = \frac{f_{y_{1:H_1}}}{f_{y_{1:H_0}}} \frac{\Lambda_{H_1}}{\Lambda_{H_0}} \]
\[ x' = 1 - \int_{-\infty}^{\infty} f_{y_1|H_0}(y_1|H_0) \, dy = \int_{-\infty}^{\infty} f_{y_1|H_0}(y_1|H_0) \, dy \]

Find \( r \) from \( L(y) \)

Receiver Operating Curve (ROC)