Given one form of the element force-displacement relationship, it is possible to obtain alternate forms through simple operations. These transformations rely only on general mathematical manipulations and the principles of equilibrium and energy. They apply not only to a single element but also to complete systems, the members of which undergo compatible deformation. They are developed here because they are particularly useful in forming the framework element stiffness matrix.

Consider the three bar truss assemblage shown in Fig. 4.5. Stiffness equations can be generated for the unsupported structure of Fig. 4.5(a) that can be expressed as

\[
\begin{bmatrix}
\{F_f\} \\
\{F_s\}
\end{bmatrix} =
\begin{bmatrix}
[k_{ff}] & [k_{fs}] \\
[k_{sf}] & [k_{ss}]
\end{bmatrix}
\begin{bmatrix}
\{\Delta_f\} \\
\{\Delta_s\}
\end{bmatrix}
\]

(4.11)

where subscript \( f \) designates components associated with the free dof and subscript \( s \) designates components associated with the support dof. Considering Fig. 4.5(b), the vectors in (4.11) are

\[
\{F_f\} = \begin{bmatrix} F_{x1} & F_{y1} & F_{x2} \end{bmatrix}^T \\
\{F_s\} = \begin{bmatrix} F_{y2} & F_{x3} & F_{y3} \end{bmatrix}^T \\
\{\Delta_f\} = \begin{bmatrix} u_1 & v_1 & u_2 \end{bmatrix}^T \\
\{\Delta_s\} = \begin{bmatrix} v_2 & u_3 & v_3 \end{bmatrix}^T
\]

(4.12)

Since the support displacements are zero, (4.11) can be rewritten as

\[
\begin{bmatrix}
\{F_f\} \\
\{F_s\}
\end{bmatrix} =
\begin{bmatrix}
[k_{ff}] \\
[k_{sf}]
\end{bmatrix}
\begin{bmatrix}
\{\Delta_f\} \\
\{\Delta_s\}
\end{bmatrix}
\]

(4.13)

The unknowns in (4.13) are the nodal displacements \( \{\Delta_f\} \). These displacements can be calculated in terms of the applied loads at the free dof by solving the upper partition of (4.13)

\[
\{\Delta_f\} = [k_{ff}]^{-1} \{F_f\} = [d] \{F_f\}
\]

(4.14)

Solving the lower partition of (4.13)

\[
\{F_s\} = [k_{sf}] \{\Delta_f\} = [k_{sf}] [d] \{F_f\}
\]

will provide the support reactions for zero support displacements.
Equation (4.14) shows that the element flexibility matrix \([d]\) equals the inverse of the stiffness matrix associated with the free displacement and force variables. Such a restriction is necessary for flexibility equations since such equations are only defined for stable structures and normally restrict attention to statically determinate structures. A stable structure is necessary since flexibility coefficient \(d_{ij}\) equals the release \(i\) displacement due to a unit virtual force at release \(j\). If the structure is not stable, then infinite, rigid body displacements will result.

This is not the case for stiffness coefficient \(k_{ij}\) since it represents the force at displacement \(dof\) \(i\) due to a unit virtual displacement at \(dof\) \(j\) with all other displacements equal to zero. Since all displacements are prescribed in the stiffness analysis, rigid body motion is prevented in determining the stiffness coefficients.

Note also that when an element or structure is supported in a statically determinate fashion, the complementary strain energy has a value that is independent of the specific form of the support system.

However, if a statically indeterminate support condition is used, then the complementary strain energy is dependent on upon the support condition. This means that the complementary strain energy is different for each such support system. Since the global analysis equations can be obtained through a summation of element strain energies (see Chapter 6, MGZ), element strain energies should be independent of local element support conditions.

Equations (4.11) – (4.14) led to the flexibility equations of (4.14) by starting with the stiffness equations and reducing the equations to a stable, statically determinate structure. Reversing the procedure and making use of equilibrium can be used to construct the complete stiffness equations of (4.11). This process begins by pre-multiplying both sides of (4.14) by \([d]^{-1}\):\[
\{F_f\} = [d]^{-1} \{\Delta_f\} = [k_{ff}]\{\Delta_f\} \quad (4.15)
\]

Recalling that the columns of the stiffness matrix are in equilibrium, the lower partition of the stiffness...
matrix \([k_{sf}]\) can be obtained as
\[
\{F_s\} = [k_{sf}]\{\Delta_f\} = [\Phi][k_{ff}]\{\Delta_f\} = [\Phi][d]^{-1}\{\Delta_f\}
\]
\[\Rightarrow [k_{sf}] = [\Phi][d]^{-1}
\] (4.22)
where \([\Phi]\) = equilibrium matrix and is generated from
\[
\{F_s\} = [\Phi]\{F_f\}
\] (4.19)

To complete the construction of (4.11), the matrices that pre-multiply \([\Delta_s]\) must be constructed, i.e., \([k_{fs}]\) and \([k_{ss}]\). Remembering that the stiffness matrix is symmetric:

\[\begin{bmatrix}
[k_{fs}] & [k_{sf}] \\
[k_{sf}] & [k_{ss}]
\end{bmatrix}
\]

Combining (4.11), (4.16), (4.22), and (4.24), the stiffness matrix can be expressed as
\[
[k] = \begin{bmatrix}
[k_{ff}] & [k_{fs}] \\
[k_{sf}] & [k_{ss}]
\end{bmatrix}
\begin{bmatrix}
[d]^{-1} & [d]^{-1}[\Phi]^T \\
[\Phi][d]^{-1} & [\Phi][d]^{-1}[\Phi]^T
\end{bmatrix}
\] (4.25)

Equation (4.25) will be used in the next section to construct the stiffness matrix for a three-dimensional frame element.

However, the best application of (4.25) is in generating “exact” stiffness equations for problems that are difficult to construct “exact” kinematic fields but for which the generation of exact equilibrium equations is fairly easy. Examples include non-prismatic beams, circular beams, flexibly connected beams, etc. These topics will be covered in subsequent chapters.
In summary, (4.25) shows that the stiffness matrix can be constructed from the inverse of the flexibility matrix \([d]\) and a matrix that derives from the element static equilibrium relationships – the equilibrium matrix \([\Phi]\). The property of symmetry was invoked in constructing \([k_{fs}]\) from \([k_{sf}]\). Equation (4.25) shows that matrix \([k_{ss}]\) is obtained through a matrix triple product in which the pre-multiplier of the central matrix is equal to the transpose of the post-multiplier matrix. This triple product is known as a congruent transformation, which produces a symmetric matrix when the central matrix is symmetric. Thus, \([k_{ss}]\) is assured to be symmetric. We have made extensive use of congruent transformations in this course.

Equation (4.25) is a general formula for transformation from flexibility to stiffness form that includes rigid body motion degrees of freedom. The number \(s\) of support forces is dictated by the requirements of stable, statically determinate support. However, the number \(f\) of free degrees of freedom is essentially limitless.

3D Framework Element Stiffness Matrix

In this section, we shall ignore deformations resulting from shear since for most frame members such deformations are small (less than 1% for sections with a length to depth ratio of 20). Furthermore, displacements resulting from out-of-plane (longitudinal) warping of a cross section that torsional forces may cause will also be ignored since such deformations are typically small for most steel members and essentially zero for all solid members. Figure 4.6 illustrates the right-hand sign convention for both the displacements and forces. The local x-axis coincides with the centroidal axis of the element. The local y-axis is taken as the weak or minor principal cross section axis (same as used in the AISC steel design manual) and the local z-axis is taken to be the strong or major principal cross section axis (note: this is the x-axis in the AISC steel design manual).
The 12 degree-of-freedom element shown in Fig. 4.6 is straight, prismatic, and symmetrical about both principal cross section axes, referred to as a bisymmetrical section. For such a member, the shear center coincides with the centroid. In addition, four distinct and uncoupled deformation modes are appropriate: axial, flexural about each principal cross section axis, and torsion. Uncoupled mode means that a particular force vector only causes a displacement in the same vector direction. For example, $F_{x1}$ and $F_{x2}$ in Fig. 4.6 can only cause axial deformation and hence only axial displacements $u_1$ and $u_2$ can be produced by these two forces. Also, the torsional moments (torques) $M_{x1}$ and $M_{x2}$ only cause rotational displacements $\theta_{x1}$ and $\theta_{x2}$ in the element. Similarly, for bending about the y-axis, only forces $F_{z1}$, $F_{z2}$, $M_{y1}$ and $M_{y2}$ cause displacements $w_1$, $w_2$, $\theta_{y1}$ and $\theta_{y2}$ in the element.

Lastly, for bending about the z-axis only forces $F_{y1}$, $F_{y2}$, $M_{z1}$ and $M_{z2}$ cause displacements $v_1$, $v_2$, $\theta_{z1}$ and $\theta_{z2}$ in the element. Uncoupled response for the bending deformation modes is known from elementary strength of materials where we learned that bending moments and shearing forces in one principal plane do not cause any deformation outside the principal plane.

In generating the stiffness matrix for the 12 dof of Fig. 4.6, we will develop four distinct submatrices: axial, torsion, bending about the z-axis, and bending about the y-axis. This will result in a 12x12 stiffness matrix that contains zeros in the locations where the various forces and displacements do not interact, i.e., are uncoupled.
Consider the axial force element shown in Fig. 4.7. The axial displacement at element node 1 in Fig. 4.7(a) is calculated as

\[ u_1 = \int_0^L \sigma \, dx = \int_0^L \frac{F_{x1}}{EA} \, dx = \frac{F_{x1}L}{EA} \]

For this case, \([d] = \frac{L}{EA}\) and \(\{F\} = F_{x1}\). Thus, \([k_{fl}] = [d]^{-1} = \frac{EA}{L}\). The equilibrium matrix is obtained from Fig. 4.7(b): \(F_{x2} = -F_{x1}\) which shows \(\{F_s\} = F_{x2}\) and \(\Phi = -1\). Substituting \([d]^{-1}\) and \(\Phi\) into (4.25):

\[
[k_{fl}] = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{4.26a}
\]

and

\[
\begin{bmatrix} F_{x1} \\ F_{x2} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{4.26b}
\]

where subscript “a” has been added to the stiffness matrix of (4.26a) to emphasize axial deformation.

Mathematically, the pure torsion element is nearly identical to the axial force element since only two forces and displacements are involved in defining the response of each mode of deformation (compare Figs. 4.7 and 4.8).
Obviously, there are differences in response since the axial force element experiences a change in element length whereas the pure torsion element twists. From strength of materials, we know that the torsional moment \( M_{x1} \) is related to the angle of twist \( \theta \) as

\[
\theta_{x1} = \int_0^L \beta \, dx = \int_0^L \frac{M_{x1}}{GJ} \, dx = \frac{M_{x1}L}{GJ}
\]

\( J = \text{torsion constant} \) \( \left(= \frac{1}{3} \sum bt^3 \text{ for an open section, } b = \text{component width and } t = \text{component thickness} ; \right) \) = polar moment of inertia for closed sections). The expression for \( \theta_{x1} \) shows that \( [d] = \frac{L}{GJ} \) and from Fig. 26.

Element Bending Stiffness Equations about z-Axis

The stresses and strains at any cross section caused by bending are directed along the x-axis of the element, vary linearly with respect to the y-axis, and are constant in the x direction for a fixed value of y. In elementary mechanics, it is shown that the strain \( e_x \) is expressed as

\[
e_x = -\frac{y}{\rho_z} = -\frac{d^2 v}{dx^2}
\]

where \( \rho_z = \text{radius of curvature about the z-axis} \), which is approximated as

\[
\frac{1}{(d^2 v/dx^2)}.
\]

\( \text{NOTE: Positive curvature } \kappa_z = 1/\rho_z \text{ causes tension in the bottom fibers and compression in the top fibers.} \)

From Hooke’s law

\[
\sigma_x = E \epsilon_x = -Ey \frac{d^2 v}{dx^2}
\]

Considering the pure bending case results in no resultant axial force, only the resultant bending moment \( M_z \):

\[
M_z = \int_A \sigma_y y \, dA = \int_A E \frac{d^2 v}{dx^2} y^2 \, dA
\]

\[
= E \frac{d^2 v}{dx^2} \int_A y^2 \, dA = EI_z \frac{d^2 v}{dx^2}
\]

where \( I_z = \text{second moment of area about the z-axis} \).
Consider the beam of Fig. 4.9. The flexibility equations are expressed as

$$ \begin{pmatrix} y_1 \\ \theta_{z1} \end{pmatrix} = [d] \begin{pmatrix} F_{y1} \\ M_{z1} \end{pmatrix} $$  (bz.1)

Calculation of the flexibility coefficients can be expressed as

$$ d_{ij} = \int_0^L \frac{M_{vi}(x)M_{vj}(x)}{EI_z} \, dx \quad (bz.2) $$

where $M_{vi}$ and $M_{vj}$ are the virtual moment equations for dof $i$ and $j$, respectively. (A curvature moment expression is included in your class notes.) For the cantilever beam of Fig. 4.9(a), the virtual moment equations are

$$ M_{v1} = x 
M_{v2} = -1 \quad (bz.3) $$

where $x$ is measured from element node 1. Substituting (bz.3) into

$$ (bz.2) $$

for $i, j = 1, 2$ and those results into (bz.1) leads to the following flexibility and inverse flexibility matrices

$$ [d] = \frac{1}{EI_z} \begin{bmatrix} L^3/3 & -L^2/2 \\ -L^2/2 & L \end{bmatrix} \quad (bz.4) $$

$$ [d]^{-1} = \frac{12EI_z}{L^3} \begin{bmatrix} 1 & L/2 \\ L/2 & L^2/3 \end{bmatrix} \quad (bz.5) $$

To complete the stiffness matrix calculations of (4.25), we need to calculate the equilibrium matrix. Expressing the support forces $\{F_s\}$ ($= <F_{y2} M_{z2}^T>$) of Fig. 4.9(b) in terms of the free forces $\{F_f\}$ ($= <F_{y1} M_{z1}^T>$) leads to

$$ \sum F_y = 0 \Rightarrow F_{y2} = -F_{y1} 
\sum M_2 = 0 \Rightarrow M_{z2} = F_{y1} L - M_{z1} $$

In matrix form:

$$ \begin{pmatrix} F_{y2} \\ M_{z2} \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ L & -1 \end{bmatrix} \begin{pmatrix} F_{y1} \\ M_{z1} \end{pmatrix} \Rightarrow [\Phi] = \begin{bmatrix} -1 & 0 \\ 0 & L \end{bmatrix} \quad (bz.6) $$
Substituting (bz.5) and (bz.6) into (4.25) gives

\[
\begin{bmatrix}
F_{y_1} \\
M_{z_1} \\
F_{y_2} \\
M_{z_2}
\end{bmatrix} = 
\begin{bmatrix}
12 & 6L & -12 & 6L \\
6L & 4L^2 & -6L & 2L^2 \\
-12 & -6L & 12 & -6L \\
6L & 2L^2 & -6L & 4L^2
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
\theta_{z_1} \\
\theta_{z_2}
\end{bmatrix}
\]

\[
[k_{bz}] = \frac{EI_z}{L^3}
\begin{bmatrix}
12 & 6L & -12 & 6L \\
6L & 4L^2 & -6L & 2L^2 \\
-12 & -6L & 12 & -6L \\
6L & 2L^2 & -6L & 4L^2
\end{bmatrix}
\]

Subscript \(bz\) emphasizes that the stiffness matrix is for bending about the z-axis.

Element Bending Stiffness Equations about \(y\)-Axis

\[
\begin{bmatrix}
F_{z_1} \\
M_{y_1} \\
F_{z_2} \\
M_{y_2}
\end{bmatrix} = 
\begin{bmatrix}
12 & -6L & -12 & -6L \\
-6L & 4L^2 & 6L & 2L^2 \\
-12 & 6L & 12 & 6L \\
-6L & 2L^2 & 6L & 4L^2
\end{bmatrix}
\begin{bmatrix}
w_1 \\
\theta_{y_1} \\
w_2 \\
\theta_{y_2}
\end{bmatrix}
\]

\[
[k_{by}] = \frac{EI_y}{L^3}
\begin{bmatrix}
12 & -6L & -12 & -6L \\
-6L & 4L^2 & 6L & 2L^2 \\
-12 & 6L & 12 & 6L \\
-6L & 2L^2 & 6L & 4L^2
\end{bmatrix}
\]

Subscript \(by\) emphasizes bending about the \(y\)-axis.

negative of the transverse displacement gradient whereas the rotational displacements about the \(z\)-axis equals the transverse displacement gradient, i.e.,

\[
\theta_y = -\frac{dw}{dx} \quad \text{and} \quad \theta_z = \frac{dv}{dx}
\]

This minor complication is unavoidable if a consistent right-hand-rule for displacements is to be adopted. Since \(y\)-axis bending is the same as \(z\)-axis bending, with the exception stated, the stiffness equations for \(y\)-axis bending can be expressed as

Fig. 4.10 \(y\)-Axis Bending Element.

Figure 4.10 shows the displacement and force variables for bending about the \(y\)-axis. Comparing this figure with Fig. 4.9 shows that the rotational displacements about the \(y\)-axis are the...
Complete Element Stiffness Equations

Combining the stiffness equations of (4.26b), (4.27), (4.32), and (4.33) into a 12 x 12 system of stiffness equations results in (4.34). Zeros in this matrix identify that the local coordinate stiffness equation (equilibrium) does not interact with the degree of freedom that is multiplied by the zero terms.

Such uncoupling results from using principal cross section coordinates and is true even for unsymmetric sections. However, for unsymmetric or singly symmetric sections, force variables $F_x$, $M_y$, and $M_z$ as well as displacement variables $u$, $\theta_y$, and $\theta_z$ are expressed in terms of the cross section centroidal axis. Force variables $F_y$, $F_z$, and $M_x$ as well as displacement variables $v$, $w$, and $\theta_x$ are expressed in terms of the shear center axis for the element. See 3D rotation notes on how to obtain all the variables at the element centroid.

Exact Nonprismatic Beam Stiffness Equations

Recall that the element stiffness matrix can be expressed in terms of the element flexibility matrix $[d]$ and equilibrium matrix $[\phi]$ as shown in (4.25). Exact stiffness equations can be generated provided the
flexibility and equilibrium matrices are exact. Considering the generic nonprismatic beam element of Fig. 1, the free and support forces vectors are

$$\{F\}_1 = < F_{x1} M_{z1} M_{z2} >^T$$  \hspace{1cm} (2a)

$$\{F\}_S = < F_{x2} F_{y1} F_{y2} >^T$$  \hspace{1cm} (2b)

An exact flexibility matrix for the element of Fig. 1 can be constructed using the principle of virtual forces (see Section 7.6 in McGuire, Gallagher and Ziemien, 2000)

$$[d] = \int_0^L [(Q)^T [C]^{-1} Q] dx$$  \hspace{1cm} (3)

where

$$[Q] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{x}{L} & 0 \\ 0 & 0 & \frac{x}{L} \end{bmatrix} = \text{force interpolation matrix (i.e., interpolation of the internal member axial force } \mathcal{F}_x \text{ and bending moment } \mathcal{M}_z \text{),}$$

$$[C]^{-1} = \begin{bmatrix} \frac{1}{EA(x)} & 0 \\ 0 & \frac{1}{EI_z(x)} \end{bmatrix} = \text{inverse material stiffness matrix or material compliance matrix; } A(x) = \text{cross section variation of area; and } I_z(x) = z\text{-axis moment of inertia variation. Evaluating the matrix triple product of (3) leads to}

$$[d] = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_{22} & d_{23} \\ 0 & d_{32} & d_{33} \end{bmatrix}$$

$$= \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & d_3 \\ 0 & d_3 & d_4 \end{bmatrix}$$  \hspace{1cm} (4)

where the unique flexibility coefficients $d_i$ are calculated as

$$d_1 = \int_0^L \frac{1}{EA(x)} dx$$

$$d_2 = \int_0^L \frac{(\frac{-x^2}{2} + x^2 \frac{1}{E} \frac{1}{I_z(x)})}{EI_z(x)} dx$$

$$d_3 = \int_0^L \frac{(\frac{-x^2}{2} + x^2 \frac{1}{E} \frac{1}{I_z(x)})}{EI_z(x)} dx$$

$$d_4 = \int_0^L \frac{x^2 \frac{1}{E} \frac{1}{I_z(x)}}{EI_z(x)} dx$$

Closed-form evaluation of (4) is not possible for arbitrary cross sections, e.g., local buckling of component plates making up a cold-form section which are dependent on the compressive stress levels. Consequently, the flexibility coefficients of (4) must be evaluated numerically. A reliable integration scheme is Labatto quadrature, in which the abscissa $(-1 \leq x_i \leq 1)$ and weighting coefficients $(A_i)$ are given in Table 1 for $N = 2$ to $N = 10$ integration points.
Table 1. Labatto Quadrature Data

<table>
<thead>
<tr>
<th>x</th>
<th>11.00000000</th>
<th>15.00000000</th>
<th>19.00000000</th>
<th>23.00000000</th>
<th>27.00000000</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.00000000</td>
<td>0.00000000</td>
<td>0.00000000</td>
<td>0.00000000</td>
<td>0.00000000</td>
</tr>
</tbody>
</table>

Once the flexibility coefficients of (4) are evaluated, the inverse flexibility matrix is simply

\[
[D]^{-1} = \begin{bmatrix}
    0 & 0 & 0 & 0 & 0 \\
    0 & d_1 & -d_2 & d_3 & d_2 \\
    0 & -d_1 & d_2 & 0 & 0 \\
    0 & d_4 & -d_4 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
k_1 & 0 & 0 & 0 & 0 \\
k_2 & 0 & k_3 & 0 & 0 \\
k_4 & 0 & 0 & k_3 & 0 \\
k_5 & 0 & 0 & 0 & k_4
\end{bmatrix} (5)
\]

where \( D = d_2 d_4 - d_3 d_3 \). The complete stiffness matrix also requires calculation of the equilibrium matrix

\[
\{F\} = \{\Phi\} \{F\}_t
\]

\[
\Rightarrow \begin{bmatrix}
    F_{x_2} \\
    F_{y_1} \\
    F_{y_2}
\end{bmatrix} = \begin{bmatrix}
    -L & 0 & 0 \\
    0 & 1 & 0 \\
    0 & -1 & -L
\end{bmatrix} \begin{bmatrix}
    F_{x_1} \\
    M_{z_1} \\
    M_{z_2}
\end{bmatrix} (6)
\]

Using the results of (5) and (6) in (4.25) results in an “exact” stiffness matrix formulation for the nonprismatic element of Fig. 1.

**Exact Fixed-End Force Vector for the Nonprismatic Beam**

Exact fixed-end forces at the free and support dof are generated as

\[
\{\Delta_t\} = \begin{bmatrix}
    \hat{u}_1 & \theta_{12} & \theta_{22}
\end{bmatrix}^T = \int_0^L \{Q\}^T \{C\}^{-1} \{P(x)\} dx (7)
\]

where \( \{P(x)\} = \{N(x) \ M_z(x)\}^T \) = internal force vector distribution for the axial stress resultant \( N(x) \) and moment stress resultant \( M_z(x) \).

**Fig. 2. Nonprismatic Beam Subjected to Uniform Load**
Consider the uniform load case depicted in Fig. 2, the axial force distribution is zero and the internal bending moment is

\[ M_z(x) = \frac{qL}{2} x - q x^2 \]

Substituting these force resultants in (7) gives

\[ u_1 = 0 \]
\[ \theta_{z1} = \frac{q}{2} \int_0^L \frac{L}{Ez} (Lx - x^2) \, dx \]
\[ \theta_{z2} = \frac{q}{2} \int_0^L \frac{L}{Ez} (Lx - x^2) \, dx \]

Using the flexibility equations

\[ [d] f_F = \Delta_F \] (9)

where for this case \( \{F_f\} = \{F_f^E\} = \) equivalent element nodal forces at the free force dof. Substituting this observation into (9) and solving

\[ \{F_f^E\} = - [d]^{-1} \Delta_F \] (10)

The fixed-end forces at the support degrees of freedom \( \{F_s^F\} \) are obtained by simply considering equilibrium

\[ \{F_s^F\} = [\Phi] \{F_f^F\} + \{\Phi_q\} \] (12)

where \( \{\Phi_q\} = qL/2 <0 1 1>^T = \)

= uniform distributed load contribution to equilibrium as shown in Fig. 2. Again, for arbitrary moment of inertia variation, the free dof displacements should be evaluated using Labatto quadrature.

Rearranging the displacements such that they are ordered in the usual fashion \( (\Delta) = <u_1 v_1 \theta_{z1} u_2 v_2 \theta_{z2}>^T \) leads to the following representation of the element stiffness matrix in terms of the unique stiffness coefficients

\[ [k] = \begin{bmatrix}
\alpha_1 & 0 & 0 & \alpha_1 & 0 & 0 \\
0 & \alpha_2 & \alpha_3 & 0 & \alpha_2 & \alpha_5 \\
0 & \alpha_3 & \alpha_4 & 0 & \alpha_3 & \alpha_6 \\
-\alpha_1 & 0 & 0 & \alpha_1 & 0 & 0 \\
0 & -\alpha_2 & -\alpha_3 & 0 & \alpha_2 & \alpha_5 \\
0 & \alpha_4 & \alpha_6 & 0 & \alpha_5 & \alpha_7 \\
\end{bmatrix} \] (13)

where \( \alpha_1 = k_1, \alpha_2 = (k_2 + 2k_3 + k_4)/L^2, \alpha_3 = (k_2 + k_3)/L, \alpha_4 = k_2, \alpha_5 = (k_3 + k_4)/L, \alpha_6 = k_3, \alpha_7 = k_4; k_1, k_2, k_3, \) and \( k_4, \) are the stiffness coefficients in (5). Notice in (13) that the rotational stiffness coefficients \( \alpha_3 \) and \( \alpha_5 \) associated with the transverse displacement dof are not equal as is the case for a prismatic element; end rotational.
stiffness coefficients $\kappa_4$ and $\kappa_7$ are not equal; and $\kappa_6 \neq \kappa_4/2$ or $\neq \kappa_7/2$.

Extension of the above concepts to three dimensions should be fairly straightforward. You need to consider torsion and bending about the weak axis separately. The equations for torsion will mirror the axial force case except that $GJ(x)$ will replace $EA(x)$. For bending about the $y$-axis (weak), replace $I_x(x)$ with $I_y(x)$ and take into account the sign change in the off-diagonal stiffness coefficients (see earlier section of these slides).