1. **Introduction**

This laboratory investigation involves the measurement of a special type of time-varying signal called a periodic signal, i.e., a signal that repeats. The system that will be demonstrated consists of a cantilever spring with an attached mass. The system will be excited by first displacing the cantilever spring, and then releasing it. The system response will be tracked using an accelerometer and digital oscilloscope. An accelerometer is a device that transforms accelerations into an analog voltage signal. Utilizing theory you will be able to predict the frequency of the signal; these predictions will then be compared to the oscilloscope readings.

2. **Educational Objectives**

The goal of this laboratory investigation is to model a simple mass-spring system using Newton’s Second Law, and then verify system response with an accelerometer. Upon completion of this laboratory exercise students will be able to:

1. Observe the use of a digital oscilloscope to capture periodic waveforms.
2. Apply formulas to describe the performance of a simple mass-spring system.
3. Calculate the natural frequency of a simple mass-spring system.

3. **Definitions**

   - **Frequency** - Number of cycles of a periodic signal completed per second, usually reported in Hertz (1 Hz = 1 cycle/sec).
   - **Period** - Time required for one complete cycle of a periodic signal.
   - **Amplitude** - Represents half of the absolute value of the maximum minus the minimum values of a periodic signal.

   **Differential Equation** - An equation consisting of terms that include a function and derivatives of that function.

4. **Applications**

It is often desirable for engineers to describe the response of a dynamic system. A dynamic system is any system that is characterized by a time varying response. For the purpose of this demonstration we will investigate the response of a simple mass-spring system. By extending this system to include additional components it is possible to simulate the suspension of a car and predict the motion of the car body when the wheels hit a pothole. Further development of mass-spring models enable engineers to predict the response of a steel building to an earthquake. Common elements to such models include simple components such as masses, springs and dashpots. By developing mathematical models of these components it is possible to describe the response of dynamic systems using differential equations. The use of differential equations is common to all disciplines of engineering.
5. Modeling a Spring-Mass System

The simple mass-spring system shown in Figure 1 will be the basis of this laboratory investigation. The spring is a cantilever beam, and the mass is fixed to the free end of the beam.

![Figure 1: Simple Mass-Spring System](image1)

To develop a mathematical model for this system, redraw the mass-spring system as shown in Figure 2. In this case the cantilever spring is replaced by a helical coil spring.

![Figure 2: Simplified Schematic Diagram](image2)

If we remove the weight from the system and replace the spring with a force vector, we now have a free-body-diagram (FBD). A free-body-diagram shows an isolated body with all of the applied and body forces represented with vectors. Figure 3 shows the FBD for our spring-mass system.

![Figure 3: FBD of Mass-Spring System](image3)
From the FBD and using Newton’s Second Law, we can write the following equation,

\[ Ma = Mg - Fs \]

where \( F_s \) is the force exerted by the spring in Newtons (N); \( Mg \) is a body force as a result of the gravitational field (N); and \( a \) is the resulting acceleration of the body. Please note that downward forces are positive. Utilizing Hooke’s Law, we can write an equation that describes the spring force. This equation is,

\[ F_s = ky \]

where \( k \) is the spring constant (N/mm). When we attach the mass to the spring, the spring undergoes a deformation where the spring force is equivalent to the gravitational force, we can now rewrite the previous equation as,

\[ F_s = k(y + y_{\text{static}}) \]

where \( y_{\text{static}} \) is the displacement of the spring that balances the gravitational force associated with the mass. Because this initial displacement of the spring balances the gravitational force, we can rewrite the equation derived using Newton’s Second Law and the FBD as,

\[ M \frac{d^2y(t)}{dt^2} = -ky(t) \]

Note that \( a \) is replaced with differential notation for \( y \). \( y \) is also replaced with \( y(t) \) to denote a time-varying function, and all of the terms with \( y \) have been moved to the left side of the equation. The resulting equation is termed a second order differential equation. To solve this equation we must turn to Calculus – the language of engineers. While it may be some time before you get to Calculus IV, and learn how to solve these types of equations, we will propose a solution, and then see if it works!

\[ M \frac{d^2y(t)}{dt^2} + ky(t) = 0 \]

Assume that one possible solution to the above equation is

\[ y(t) = A\cos(2\pi ft) + B\sin(2\pi ft) \]

The solution \( y(t) \) implies that \( y \) is a function of time – that is, \( y \) varies with time. We do not know the values of \( A \) and \( B \). To solve for \( A \) and \( B \) we must look at the initial conditions.

Prior to looking at the initial conditions we must first differentiate the solution. The first derivative is,

\[ \frac{dy(t)}{dt} = -2\pi fA\sin(2\pi ft) + 2\pi fB\cos(2\pi ft) \]
Recall that,
\[ \frac{d}{dx} \cos(ax) = -a \sin(ax) \]
\[ \frac{d}{dx} \sin(ax) = a \cos(ax) \]

Differentiating the previous function a second time we have the following,
\[ \frac{d^2 y}{dt^2} = -4\pi^2 f^2 A \cos(2\pi ft) - 4\pi^2 f^2 B \sin(2\pi ft) \]

Another handy trick is to make a substitution for frequency. While it may not be obvious at first, the following substitution simplifies our efforts.
\[ f = \frac{1}{2\pi} \sqrt{\frac{k}{M}} \]

With this substitution, the second derivative of \( y(t) \) becomes
\[ \frac{d^2 y(t)}{dt^2} = -\frac{k}{M} A \cos(\sqrt{\frac{k}{M}} t) - \frac{k}{M} B \sin(\sqrt{\frac{k}{M}} t) \]

It should be clear at this point that our assumed solution is valid. However, we must evaluate the constants \( A \) and \( B \). To do this we will look at the initial conditions. At \( t = 0 \) we know that the position is \( y_i \) (the position of our initial displacement) and therefore \( A \) must be equal to \( y_i \). If we look at the first derivative of \( y(t) \), the velocity at \( t = 0 \), we find that \( B = 0 \). The final solution becomes:
\[ y(t) = y_i \cos(\sqrt{\frac{k}{M}} t) \]

At last – a valid solution! Also remember that the frequency of the excited system will follow the substitution that we made above:
\[ f = \frac{1}{2\pi} \sqrt{\frac{k}{M}} \]

The only thing left to do is prove this analysis is appropriate by collecting some system response data. We accomplish this by using a digital oscilloscope and an accelerometer.

6. Accelerometers

Accelerometers are designed to measure acceleration. This measurement is accomplished by suspending a mass within a case. The force between the case and the mass is measured with a strain gage which is a resistive element. As this element is stretched, its resistance changes linearly. This change in resistance is proportional to acceleration. We must calibrate the accelerometer to compare the acceleration to the output voltage from this sensor.
7. Example Calculations

There are several steps involved in solving a vibrating beam-mass system such as the one used in this demonstration. The following example is one way to find the spring constant and natural frequency of the system.

Example: A steel beam is cantilevered (i.e., it is fixed at one end), at the free end of this beam there is a block that is 500 mm from the fixed end. This block has a mass of 1.0 kg. The beam is 50.0 mm wide with a thickness of 1.91 mm.

The first step is to find the moment of inertia of the beam ($I$). For this you will need the following equation:

$$ I = \frac{1}{12}wt^3 $$

where: $I$ is the second moment of the cross-sectional area ($m^4$); $w$ is the width of the beam (m); and $t$ is thickness of the beam (m).

$$ I = \frac{1}{12}(0.05m)(1.91\times10^{-3}m)^3 = 2.903\times10^{-11}m^4 $$

The next step is to find the spring constant, $k$, using the following equation.

$$ k = \frac{3EI}{l^3} $$

where: $k$ is the spring constant (N/m); $E$ is the modulus of elasticity for the spring material ($20\times10^{10}$ N/m$^2$); $I$ is the second moment of the cross-sectional area ($m^4$); and $l$ is the length of the cantilever beam (m),

$$ k = \left[ \frac{3(20\times10^{10} N/m^2)(2.903\times10^{-11} m^4)}{(0.5m)^3} \right] = 139.357 \frac{N}{m} $$
The final step is to determine the natural frequency of the system:

\[
f_n = \left(\frac{1}{2\pi}\right) \sqrt{\frac{3EI}{Ml^4}}
\]

If you recall the equation for finding the spring constant, a substitution can be made:

\[
f_n = \left(\frac{1}{2\pi}\right) \sqrt{\frac{k}{M}}
\]

where: \(f_n\) is the natural frequency (Hz); \(k\) the spring constant (N/m); and \(M\) is the mass of the block (kg). Solving for the natural frequency,

\[
f_n = \left(\frac{1}{2\pi}\right) \sqrt{\frac{139.357 \text{ N/m}}{1.0 \text{ kg}}} = 1.879 \text{ Hz}
\]

This means that the system will vibrate at a frequency of 1.879 Hz if excited with a step input.

8. **Laboratory Procedures** - (Instructor will demonstrate experimental set-up)

I. **Spring Constant**
   1. Using a spring steel beam of the following size:
      
      Width = 3.8 cm  
      Thickness = 0.062 in
   2. The beam will be set at two lengths for this series of lab experiments. These lengths will be 30 cm and 45 cm from the clamped edge of the beam to the center of the mass.
   3. **Calculate the two spring constants, i.e., the spring constant of a 30 cm length beam and the spring constant of a 45 cm length beam. Make sure to have your numbers in meters, NOT in millimeters. Show your work and circle your answers.**

II. **System Natural Frequency**
   1. We will use a series of cylindrical brass masses with a bolt and wing nut to determine the systems natural frequency. This data is provided on your data sheet.

\( M_{\text{acc \& bolt \& nut}} = 31.5 \text{ g} \) (This gets added to the weight)

2. Repeat this experiment using 4 different combinations of weight and length. The combinations are as follows:
   a- A 30 cm beam with a 2 Newton weight at the end  
   b- A 30 cm beam with a 3 Newton weight at the end  
   c- A 45 cm beam with a 2 Newton weight at the end  
   d- A 45 cm beam with a 3 Newton weight at the end
3. **Compute the natural frequency for each condition. Show your work on data sheet and circle your answers.**
   a- A 30 cm beam with a 2 Newton weight at the end
   b- A 30 cm beam with a 3 Newton weight at the end
   c- A 45 cm beam with a 2 Newton weight at the end
   d- A 45 cm beam with a 3 Newton weight at the end

![Figure 5: Cantilever Spring-Mass System with Accelerometer](image)

**III. Step Input Test**

1. The following steps explain how to find experimentally the natural frequency of the beams. We are providing the data to you, but this describes the steps involved in collecting that data.
2. Apparatus set-up: For rigidity, we use a 1 cm thick steel plate that is 3.8 cm wide and 10 cm long to clamp the spring steel beam to the table top with a C-clamp. This should be done so that the beam measures 30 cm from the mass location to the edge of the table with the 1 cm steel plate being even with the edge of the table.
3. Attach the mass to the beam using the bolt and wing nut. Run the bolt through the hole in the end of the beam from underneath.
4. Affix the accelerometer to the beam with the hole nearest to the table and turn on the oscilloscope.

![Figure 6: Digital Oscilloscope](image)
5. Now press the SAVE/RECALL button on the upper right hand side of the oscilloscope. On the screen you will see the option to “Recall Factory Setup”. Select this option and then press the “OK Confirm Factory Init” button and then the “MENU OFF” button. Next, press the “MEASURE” button and select the oscilloscope settings to measure Frequency, Max, and Min and press the “MENU OFF” button. Note with the beam at rest the Max and Min voltages will be nearly the same.

6. Set the offset voltage on the oscilloscope. To do this first press the “MENU” button that is next to the white “REF” button. There will be a list of options at the bottom of the screen. Select the “Offset” option. At the very top of the oscilloscope, directly in line with the “MENU” button there is a round knob. Adjust this knob until the offset voltage is equal to the average of the Max and Min voltages and press the “MENU OFF” button. Note that the oscilloscope readout is now centered on the screen.

7. Adjust the time scale of the oscilloscope to 400 ms. This is performed by adjusting the HORIZONTAL SCALE knob.

8. We use a specialized data acquisition box to attach the accelerometer to the oscilloscope in order to take the measurements.

9. Displace the free end of the mass beam assembly 5 to 7 cm and release. Record the values for Max and Min voltage and the frequency. Repeat for each combination of weight and length. This data is provided in your data sheet.

   a- A 30 cm beam with a 2 Newton weight at the end
       Max  = 284.0 mV
       Min  = -260.0 mV
       Freq. = 5.165 Hz

   b- A 30 cm beam with a 3 Newton weight at the end
       Max  = 180.0 mV
       Min  = -196.0 mV
       Freq. = 4.53 Hz

   c- A 45 cm beam with a 2 Newton weight at the end
       Max  = 208.0 mV
       Min  = -200.0 mV
       Freq. = 2.722 Hz

   d- A 45 cm beam with a 3 Newton weight at the end
       Max  = 224.0 mV
       Min  = -220.0 mV
       Freq. = 2.38 Hz
Home Work Section:

IV. Accelerations

1. Using the fact that the accelerometers have an output of 9.6 mV/m/s² (this means that if the accelerometer was being accelerated at 1 m/s² the oscilloscope would read 9.6 mV) and the Max and Min voltages of each trial, determine the respective maximum and minimum accelerations experienced by the accelerometer. Show your work only for the 30 cm and 2 Newton trial.

   a- A 30 cm beam with a 2 Newton weight at the end
      Max =
      Min =

   b- A 30 cm beam with a 3 Newton weight at the end
      Max =
      Min =

   c- A 45 cm beam with a 2 Newton weight at the end
      Max =
      Min =

   d- A 45 cm beam with a 3 Newton weight at the end
      Max =
      Min =

V. Concluding Questions

1. In the four different combinations of weight and length how did the measured value of natural frequency compare to the calculated value in a percent error basis?

   \[
   \%\text{error} = \left(\frac{\text{measured} - \text{calculated}}{\text{measured}}\right) \times 100\%
   \]

   a- A 30 cm beam with a 2 Newton weight at the end
      Error =

   b- A 30 cm beam with all of the brass cylinders at the end
      Error =

   c- A 45 cm beam with a 2 Newton weight at the end
      Error =

   d- A 45 cm beam with all of the brass cylinders at the end
      Error =
2. Were the errors consistent for the four trials? Comment on the possible causes of error.

3. For the first trial (length of 30 cm and 2 Newton weight), how many seconds would it take for the beam to make 2000 cycles at its natural frequency? Show your work.

4. What would you expect to happen if the beam were excited at its natural frequency?

5. Compute the average acceleration of the four trials by averaging the Min and Max accelerations in part IV.

   a- A 30 cm beam with a 2 Newton weight at the end
      \[ a_{\text{average}} = \text{_________} \]
   b- A 30 cm beam with a 3 Newton weight at the end
      \[ a_{\text{average}} = \text{_________} \]
   c- A 45 cm beam with a 2 Newton weight at the end
      \[ a_{\text{average}} = \text{_________} \]
   d- A 45 cm beam with a 3 Newton weight at the end
      \[ a_{\text{average}} = \text{_________} \]

Do the numbers make sense?

What value would you expect them to be? Explain why.
Part I. Spring Constant

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<th>thickness (m)</th>
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<th>(cm)</th>
<th>(m)</th>
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Problem 3: Show your work

Part II. System Natural Frequency

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<table>
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<th>2 N</th>
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<td>30 cm</td>
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Max. (mV) | Min. (mV) | Freq. (Hz) |
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Part III Step Input Test

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Max. (mV) | Min. (mV) | Freq. (Hz) |
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Part IV Accelerations

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Max | Min

Discussion:

Part V - Answer "Concluding Questions" from Handout