EE611
Deterministic Systems

Stability
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Bounded-Input Bounded-Output

A LTI relaxed single-input single-output (SISO) system is bounded-input bounded-output (BIBO) stable iff its impulse response $g(t)$ is absolutely integrable on interval $[0, \infty)$.

$$\int_{0}^{\infty} |g(t)| \, dt \leq M < \infty$$

Bounded input $u(t)$ implies

$$|u(t)| \leq u_m < \infty \quad \forall \, t \geq 0$$

Bounded output $y(t)$ implies

$$|y(t)| \leq y_m < \infty \quad \forall \, t \geq 0$$
Examples

Determine the BIBO stability of the following systems

\[
\dot{x} = \begin{bmatrix} -2 & 6 \\ 0 & 3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)
\]

\[
y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x + \begin{bmatrix} -2 \end{bmatrix} u(t)
\]

\[
\dot{x} = \begin{bmatrix} -2 & 6 \\ 0 & 3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)
\]

\[
y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x
\]
BIBO Steady-State

Given a BIBO stable system with impulse response $g(t)$ with input $u(t)$ and output $y(t)$ for $t \geq 0$, then as $t \to \infty$

For $u(t) = a$ (constant), $y(t) \to a \hat{g}(0)$

For $u(t) = \sin(\omega_o t)$, $y(t) \to |\hat{g}(j\omega_o)|\sin(\omega_o t + \mathcal{P}(\hat{g}(j\omega_o)))$

where $\hat{g}(s)$ is the Laplace transform of the impulse response:

$$\hat{g}(s) = \int_0^\infty g(\tau) \exp(-s\tau) d\tau$$
A LTI relaxed SISO system with a rational polynomial transfer function $\hat{g}(s)$ is BIBO stable iff the real part of every pole of $\hat{g}(s)$ is less than zero.
MIMO Systems

The previous stability rules for SISO systems can be applied to multiple-input multiple-output (MIMO) systems.

BIBO stability based on absolute integrability of the impulse response, for MIMO systems requires that every element of the impulse response matrix be absolutely integrable.

BIBO stability for systems described with rational polynomials based on system poles, for MIMO systems requires that the poles of every element of the transfer matrix have real parts less than 0.
State Equations and BIBO Stability

Given a system described by the LTI state-space equations:

\[
\dot{x}(t) = Ax(t) + Bu(t) \\
y(t) = Cx(t) + Du(t)
\]

the system is BIBO stable, if the real part of the eigenvalues of system matrix A are less than 0. In some cases the eigenvalues can have real parts positive for BIBO stability as long as these modes cancel out as unstable poles in the process of converting the state-space representation to a transfer function.
BIBO for Discrete Systems

A discrete time SISO system is BIBO stable iff its impulse response $g[k]$ is absolutely summable on interval $[0 \infty)$. 

$$\sum_{k=0}^{\infty} |g[k]| \leq M < \infty$$

Bounded input $u[k]$ implies 

$$|u[k]| \leq u_m < \infty \quad \forall k \geq 0$$

Bounded output $y[t]$ implies 

$$|y[k]| \leq y_m < \infty \quad \forall k \geq 0$$
BIBO Steady-State

Given a BIBO stable system with impulse response $g[k]$ with input $u[k]$ and output $y[k]$ for $k \geq 0$, then as $k \to \infty$

For $u[k] = a$ (constant), $y[k] \to a \hat{g}(1)$

For $u[k] = \sin(\omega_0 k)$, $y[k] \to |\hat{g}(e^{j\omega_0})|\sin(\omega_0 k + \angle(\hat{g}(e^{j\omega_0})))$

where $\hat{g}(z)$ is the $z$-transform of the impulse response:

$$\hat{g}(z) = \sum_{m=0}^{\infty} g[m] z^{-m}$$
A LTI discrete-time relaxed SISO system with a rational polynomial transfer function $\hat{g}(z)$ is BIBO stable iff the magnitude of every pole of $\hat{g}(z)$ is less than 1.
MIMO Discrete Systems

The previous stability rules for SISO systems can be applied to multiple-input multiple-output (MIMO) systems.

BIBO stability based on absolute summability of the impulse response, for MIMO systems requires that every element of the impulse response matrix must be absolutely summable.

BIBO stability for systems with rational polynomial transfer functions based on system poles, for MIMO systems requires that poles for every element of the transfer matrix must have magnitudes less than 1.
Discrete State Equations and BIBO Stability

Given a system described by the LTI state-space equations:

\[ x[k + 1] = A \cdot x[k] + B \cdot u[k] \]
\[ y[k] = C \cdot x[k] + D \cdot u[k] \]

the system is BIBO stable, if the magnitudes of the eigenvalues of system matrix \( A \) are less than 1. In some cases the eigenvalues can have magnitudes greater than 1 for BIBO stability as long as these modes cancel out as unstable poles in the process of converting the state-space representation to a transfer function.
Internal Stability

Consider the zero-input response for $t \geq t_0$ for initial state $x_0 = x(t_0)$

$$\dot{x}(t) = Ax(t)$$

with solution

$$x(t) = \exp(At)x_0$$

The zero-input response is *marginally stable* (Lyapunov stable) if every finite $x_0$ excites a bounded response.

The zero-input response is *asymptotically stable* if every finite $x_0$ excites a bounded response that approaches 0 as $t \to \infty$. 
Eigenvalues and Internal Stability

A zero-input system $\dot{x} = A x$ is *marginally stable* iff the real parts of the eigenvalues of $A$ are less than or equal to zero, and for eigenvalues with real parts equal to zero the eigenvalues must belong to simple roots of the minimal polynomial of $A$.

The zero-input system $\dot{x} = A x$ is *asymptotically stable* iff the real parts of the eigenvalues of $A$ are less than zero.
Stability from Systems Matrix

Determine the stability from the Jordan form matrices below. Can BIBO stability be determined from the information below?

\[
A = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 \\
0 & 0 & -0.5 + j2 & 0 \\
0 & 0 & 0 & -0.5 - j2 \\
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -0.5 & 1 \\
0 & 0 & 0 & -0.5 \\
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -0.5 & 1 \\
0 & 0 & 0 & -0.5 \\
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -8 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -8 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -8 & 0 & 0 \\
0 & 0 & -2 & 0 \\
0 & 0 & 0 & -10 \\
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 2 \\
\end{bmatrix}
\]
Consider the zero-input response for $k \geq k_0$ for initial state $x_0 = x[k_0]$

$$x[k + 1] = Ax[k]$$

with solution

$$x[k] = A^k x_0$$

The zero-input response is *marginally stable* (Lyapunov stable) if every finite $x_0$ excites a bounded response.

The zero-input response is *asymptotically stable* if every finite $x_0$ excites a bounded response that approaches 0 as $k \to \infty$. 
Examples

Determine the internal stability of the following discrete system

\[
x[k+1] = \begin{bmatrix} -1 & 0.75 \\ 1 & 0 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[k]
\]

\[
y[k] = \begin{bmatrix} 1 & 1 \end{bmatrix} x[k]
\]
Discrete Eigenvalues and Stability

A zero-input system \( x[k + 1] = Ax[k] \) is marginally stable iff the magnitude of the eigenvalues of \( A \) are less than or equal to 1, and for eigenvalues with magnitudes equal to 1 they must belong to simple roots of the minimal polynomial of \( A \).

The zero-input system \( x[k + 1] = Ax[k] \) is asymptotically stable iff the magnitudes of the eigenvalues of \( A \) are less than 1.
Stability from Systems Matrix

Determine the stability from the Jordan form matrices below. Can BIBO stability be determined from the information below?

\[ A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -0.2 & 0 & 0 & 0 \\ 0 & 0 & -0.5 + j0.2 & 0 & 0 \\ 0 & 0 & 0 & -0.5 - j0.2 & 0 \end{bmatrix} \]

\[ A = \begin{bmatrix} 1 \times 0.5 \pi & 0 & 0 & 0 \\ 0 & 1 \times -0.5 \pi & 0 & 0 \\ 0 & 0 & -0.5 & 1 \\ 0 & 0 & 0 & -0.5 \end{bmatrix} \]

\[ A = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -0.5 & 1 \\ 0 & 0 & 0 & -0.5 \end{bmatrix} \]

\[ A = \begin{bmatrix} 1 \times 0.1 \pi & 1 & 0 & 0 \\ 0 & 1 \times 0.1 \pi & 0 & 0 \\ 0 & 0 & 1 \times -0.1 \pi & 1 \\ 0 & 0 & 0 & 1 \times -0.1 \pi \end{bmatrix} \]

\[ A = \begin{bmatrix} 1 \times 0.1 \pi & 0 & 0 & 0 \\ 0 & 1 \times 0.1 \pi & 0 & 0 \\ 0 & 0 & 1 \times -0.1 \pi & 0 \\ 0 & 0 & 0 & 1 \times -0.1 \pi \end{bmatrix} \]
Lyapunov Theorem (continuous)

All eigenvalues of $A$ have negative real parts iff for any positive definite symmetric matrix $N$, a unique symmetric and positive definite solution $M$ exists for the Lyapunov equation:

$$A' M + MA = -N$$

Note: All symmetric matrices can be diagonalized with all real values along the diagonal.

A matrix $M$ is *positive semidefinite* if $x'Mx \geq 0$ for every nonzero $x$, it is *positive definite* if the equality does not hold.
Lyapunov Theorem (discrete)

All eigenvalues of $A$ have magnitudes less than one iff for any positive definite symmetric matrix $N$, a unique symmetric and positive definite solution $M$ exists for the Lyapunov equation:

$$M - A'MA = N$$