Transient Response for Second-Order Circuits

Characteristics Equations, Overdamped-, Underdamped-, and Critically Damped Circuits
Second-Order Circuits:

In previous work, circuits were limited to one energy storage element, which resulted in first-order differential equations. Now, a second independent energy storage element will be added to the circuits to result in second order differential equations:

\[ y(t) = \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_2 x \]
Example

Find the differential equation for the circuit below in terms of $v_c$ and also terms of $i_L$

Show:

$$v_s(t) = LC \frac{d^2 v_c}{dt^2} + RC \frac{dv_c}{dt} + v_c \quad \rightarrow \quad \frac{v_s(t)}{LC} = \frac{d^2 v_c}{dt^2} + \frac{R}{L} \frac{dv_c}{dt} + \frac{1}{LC} v_c$$

$$v_s(t) = L \frac{di_L}{dt} + Ri_L + \frac{1}{C} \int_{-\infty}^{t} i_L(\tau)d\tau \quad \rightarrow \quad \frac{v_s(t)}{L} = \frac{di_L}{dt} + \frac{R}{L} i_L + \frac{1}{LC} \int_{-\infty}^{t} i_L(\tau)d\tau$$
Find the differential equation for the circuit below in terms of $v_c$ and also terms of $i_L$

Show:

$$i_s(t) = LC \frac{d^2 i_L}{dt^2} + \frac{L}{R} \frac{di_L}{dt} + i_L \quad \Rightarrow \quad \frac{i_s(t)}{LC} = \frac{d^2 i_L}{dt^2} + \frac{1}{RC} \frac{di_L}{dt} + \frac{1}{LC} i_L$$

$$i_s(t) = C \frac{dv_c}{dt} + \frac{1}{R} v_c + \frac{1}{L} \int_{-\infty}^{t} v_c(\tau) d\tau \quad \Rightarrow \quad \frac{i_s(t)}{C} = \frac{dv_c}{dt} + \frac{1}{RC} v_c + \frac{1}{LC} \int_{-\infty}^{t} v_c(\tau) d\tau$$
Solving Second-Order Systems:

- The method for determining the forced solution is the same for both first and second order circuits. The new aspects in solving a second order circuit are the possible forms of natural solutions and the requirement for two independent initial conditions to resolve the unknown coefficients.

- In general the natural response of a second-order system will be of the form:

\[ x(t) = K_1 t^m \exp(-s_1 t) + K_2 \exp(-s_2 t) \]
Natural Solutions

- Find characteristic equation from homogeneous equation:
  \[ 0 = \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_2 x \]

- Convert to polynomial by the following substitution:
  \[ s^n = \frac{d^n x}{dt^n} \quad \text{to obtain} \quad 0 = s^2 + a_1 s + a_2 \]

- Based on the roots of the characteristic equation, the natural solution will take on one of three particular forms. Roots given by:
  \[ s_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \]
Overdamped

- If roots are real and distinct \((a_1^2 - 4a_2 > 0)\), natural solution becomes

\[x_n(t) = a_1 \exp(s_1t) + a_2 \exp(s_2t)\]
Critically Damped

- If roots are real and repeated \( a_1^2 - 4a_2 = 0 \rightarrow s_1 = s_2 \),
  natural solution becomes

\[
x_n(t) = a_1 \exp(s_1 t) + a_2 t \exp(s_1 t)
\]

\[
a_1 = -3 \quad \text{and} \quad a_1 = 2
\]
If roots are complex \((a_1^2 - 4a_2 < 0)\), natural solution becomes: \((s_{1,2} = \alpha \pm j\beta)\)

\[
x_n(t) = \exp(\alpha t)[c_1 \cos(\beta t) + c_2 \sin(\beta t)]
\]

or

\[
x_n(t) = A \exp(\alpha t) \cos(\beta t + \theta)
\]

\[
A = \sqrt{c_1^2 + c_2^2}
\]

\[
\theta = -\tan^{-1}\left(\frac{c_2}{c_1}\right)
\]

\[
c_1 = A \cos(\theta) \quad c_2 = -A \sin(\theta)
\]
Example

Find the unit step response for \( v_c \) and \( i_L \) for the circuit below when:

- a) \( R=16\Omega, L=2\text{H}, C=1/24\text{ F} \)
- b) \( R=10\Omega, L=1/4\text{H}, C=1/100\text{ F} \)
- c) \( R=2\Omega, L=1/3\text{H}, C=1/6\text{ F} \)

Show:

- a) \( v_c(t) = \left(1 + \frac{1}{2}\exp(-6t) - \frac{3}{2}\exp(-2t)\right)u(t) \) \( i_L(t) = \frac{1}{8}(\exp(-2t) - \exp(-6t))u(t) \)
- b) \( v_c(t) = (1 - \exp(-20t) - 20t \exp(-20t))u(t) \) \( i_L(t) = (4 \exp(-20t))u(t) \)
- c) \( v_c(t) = (1 - \exp(-3t)(\cos(3t) + \sin(3t)))u(t) \) \( i_L(t) = (\exp(-3t) \sin(3t))u(t) \)