**DESIGN PROBLEM:** Design UE, ESLA broadside array with \( N=5 \) and smallest half-power beam width \( HP \).

1. Given \( SLL=-20\text{dB} \), find \( d \) and calculate \( HP \).
2. Given \( SLL=-10\text{dB} \), find \( d \) and calculate \( HP \).

**SOLUTION**

For both cases, we need to know what is the maximum sidelobe level (not including the grating lobes)? This has be found to be -12.04dB in the previous example.

(1) For \( SLL=-20\text{dB} \), or \( SLL=0.1 \), we can only include part of the first sidelobes.

Set \[
\left| \frac{\sin(2.5\psi_s)}{5\sin(0.5\psi_s)} \right| = 0.1, \quad \psi_s = 1.3867
\]

Since \( \psi_s = \beta d \), \( d = 1.3867\lambda/(2\pi) = 0.22\lambda \)

To find out \( HP \), let

\[
\left| \frac{\sin(2.5\psi_H)}{5\sin(0.5\psi_H)} \right| = 0.7071, \quad \psi_H = 0.5665
\]

Hence \( \psi_H = \beta d \cos \theta_R = 0.5665 \Rightarrow \theta_R = 65.9^\circ \)

\( HP = 2 \times (90 - 65.9) = 48.2^\circ \)

(2) Since the highest sidelobe level is -12.04dB, we can include part of the grating lobe in the visible region to achieve smallest \( HP \).

Let \[
\left| \frac{\sin(2.5\psi_s)}{5\sin(0.5\psi_s)} \right| = 0.3162, \quad \psi_s = 5.345
\]

Since \( \psi_s = \beta d \), \( d = 5.345\lambda/(2\pi) = 0.8507\lambda \)

From the above, \( \psi_H = 0.5665 = \beta d \cos \theta_R \Rightarrow \theta_R = 83.9^\circ \)

\( HP = 2 \times (90 - 83.9) = 12.2^\circ \)
Comments:

Consider a broadside array with N=5, SLL=-20dB, and 
\[ d = 0.7814\lambda \]. The excitations are:
\[ 1.40 : 2.25 : 2.70 : 2.25 : 1.40 \]
Using a computer program we found that the HP is \( 15.1^\circ \)
which is less than the result of \( 48.2^\circ \) in case (1) of the
previous example.

Again, if we have N=5, SLL=-10dB, and \( d = 0.86\lambda \). The
excitations are: \[ 0.746 : 0.541 : 0.590 : 0.541 : 0.746 \]

The HP is found to be \( 11.36^\circ \). This is less than the result of
\( 12.2^\circ \) obtained in the case 2 of the previous example.

It is clear that for given N and SLL, the non-uniform
excitation can achieve smaller HP compared to UE.

Question: Is there a method to design non-UE, ESLA for
narrowest beam-width given the SLL and N?

Answer: yes, use the Chebyshev synthesis method.
Chebyshev Polynomials and their properties

\[ T_0(x) = 1, \quad T_1(x) = x \]
\[ T_2(x) = x^2 - 1, \quad T_3(x) = 4x^3 - 3x \]
\[ T_4(x) = 8x^4 - 8x^2 + 1 \]

Generally, for \( n > 1 \): we have recursive expression:
\[ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \]

Closed form expression:
\[
T_n(x) = \begin{cases} 
(-1)^n \cosh\left[n \cosh^{-1}|x|\right], & x < -1 \\
\cos\left[n \cos^{-1}x\right], & -1 \leq x \leq 1 \\
\cosh\left[n \cosh^{-1}x\right], & x > 1 
\end{cases}
\]

Properties:
1. \(|T_n(x)| \leq 1 \text{ for } |x| \leq 1\)
2. All polynomial pass through \((1,1)\).
(4) Consider the symmetric excited broadside ESLA shown below. Let $\psi = \beta d \cos \theta$

For $N=2$:

$$AF = I_{-1}e^{j\psi/2} + I_1e^{-j\psi/2} = (2I_1)\cos(\psi/2)$$

For $N=3$:

$$AF = I_{-1}e^{j\psi} + I_0 + I_1e^{-j\psi} = I_0 + (2I_1)\cos(\psi)$$

$$= I_0 + (2I_1)[2\cos^2(\psi/2) - 1]$$

$$= (I_0 - 2I_1) + (4I_1)\cos^2(\psi/2)$$

For $N=4$:

$$AF = I_{-2}e^{j1.5\psi} + I_{-1}e^{j0.5\psi} + I_1e^{-j0.5\psi} + I_2e^{-j1.5\psi}$$

$$= (2I_1)\cos(\psi/2) + (2I_2)\cos(3\psi/2)$$

$$= (2I_1)\cos(\psi/2) + (2I_2)[4\cos^3(\psi/2) - 3\cos(\psi/2)]$$

$$= (2I_1 - 6I_2)\cos(\psi/2) + (8I_2)\cos^3(\psi/2)$$

This process can be continued for any value of $N$. We observed that the AF are polynomials of $\cos(\psi/2)$.
Let $x_0 > 1$. If we map $x_0$ to main beam maximum ($\theta = 90^\circ$) and map $-1 \leq x \leq 1$ to the side lobe regions, then we can achieve design Chebyshev arrays. The above mentioned mapping can be realized if we let

$$x = x_0 \cos \left( \psi / 2 \right)$$

Under this transformation, the AF for $N=4$ can be written as

$$AF = \left[ \frac{2I_1 - 6I_2}{x_0} \right] x^2 + \left[ \frac{8I_2}{x_0^3} \right] x^3 = 4x^3 - 3x$$

Where

$$8I_2 = 4x_0^3, \quad 6I_2 - 2I_1 = 3x_0$$

When $\theta = 90^\circ, \psi = 0$, and $x = x_0$, $AF(90^\circ) = T_{p^{-1}}(x_0)$. As a result, $x_0$ can be determined by main-beam maximum. Since the sidelobe value in this case are all equal to 1, the sidelobe level can be written as

$$SLL = 20 \log \left( 1 / R \right) = -20 \log(R),$$

where $R$ is the mainbeam maximum.
Design procedure: Given N, $\alpha = 0$, and SLL (dB)

1. Determine R from SLL (dB): $R = 10^{\frac{SLL}{20}}$
2. Determine $x_0$ using
   $$T_{p-1}(x_0) = \cosh \left[ (P-1) \cosh^{-1}(x_0) \right] = R$$
   $$x_0 = \cosh \left[ \frac{1}{P-1} \cosh^{-1}(R) \right]$$
3. Determine the excitation amplitudes
4. Calculate $d$ by imposing minimum beamwidth condition.

$$d = \lambda \left[ 1 - \frac{1}{\pi} \cosh^{-1} \left( \frac{1}{\gamma} \right) \right],$$

$$\gamma = \cosh \left[ \frac{1}{P-1} \ln \left( R + \sqrt{R^2 - 1} \right) \right]$$

Coefficients relationship with $x_0$:

N=2: \hspace{1cm} 2I_1 = x_0

N=3: \hspace{1cm}
\[\begin{align*}
4I_1 &= 2x_0^2 \\
2I_1 - I_0 &= 1
\end{align*}\]

\[\begin{align*}
I_1 &= x_0^2/2 \\
I_0 &= x_0^2 - 1
\end{align*}\]

N=4: \hspace{1cm}
\[\begin{align*}
8I_2 &= 4x_0^3, \\
6I_2 - 2I_1 &= 3x_0
\end{align*}\]

\[\begin{align*}
I_2 &= x_0^3/2 \\
I_1 &= (3x_0^3 - 3x_0)/2
\end{align*}\]

N=5: \hspace{1cm}
\[\begin{align*}
16I_2 &= 8x_0^4 \\
16I_2 - 4I_1 &= 8x_0^2 \\
I_0 - 2I_1 + 2I_2 &= 1
\end{align*}\]

\[\begin{align*}
I_2 &= x_0^4/2 \\
I_1 &= 2x_0^4 - 2x_0^2 \\
I_0 &= 3x_0^4 - 4x_0^2 + 1
\end{align*}\]
Design examples: N=5, Broadside, SLL=-20dB

Solution:

\[ R = 10^{\frac{SLL}{20}} = 10^{-\frac{20}{20}} = 0.1 \]

\[ x_0 = \cosh \left[ \frac{1}{4} \cosh^{-1}(10) \right] = 1.2933 \]

\[ I_2 = \frac{x_0^4}{2} = \frac{1.2933^4}{2} = 1.40 \]

\[ I_1 = 2x_0^4 - 2x_0^2 = 2.25 \]

\[ I_0 = 3x_0^4 - 4x_0^2 + 1 = 2.702 \]

Calculate separation d:

\[ \gamma = \cosh \left[ \frac{1}{4} \ln \left( 10 + \sqrt{10^2 - 1} \right) \right] = 1.2933 \]

\[ d = \lambda \left[ 1 - \frac{1}{\pi} \cos^{-1} \left( \frac{1}{1.2933} \right) \right] = 0.7814 \lambda \]

\[ x_H = \cosh \left[ \frac{1}{4} \cosh^{-1} \left( \frac{10}{\sqrt{2}} \right) \right] = 1.2266 \]

\[ \psi_H = 2 \cos^{-1} \left( \frac{x_H}{x_0} \right) = 0.465 = \beta d \cos \theta_H \]

\[ \theta_H = 82.45^0, \quad HP = 15.1^0 \]