Exact Solution to Burgers’ Equation Exhibiting Erratic Turbulent-Like Behavior

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ABSTRACT

We present an exact solution to a one-dimensional Burgers’ equation which exhibits erratic turbulent-like behavior. We compare the time series of this exact solution with a physical experimental one obtained by 1-D hot wire anemometry (HWA) for flow over a turbulator inside a channel; both have very similar erratic features. It is proposed that this type of Burgers’ equation model will provide a good tool for testing numerical algorithms for use in turbulence simulations. In addition, we have calculated the magnitudes of derivatives of the erratic solution analytically. These results demonstrate that the high-order truncation error of Taylor expansions of a nonsmooth function could be very large, implying that use of high-order difference schemes in turbulence simulation may fail to produce expected results.

1. INTRODUCTION

Burgers’ equation is a one-dimensional (1-D) analogue of the Navier-Stokes (N.–S.) equations. Due to the fact that it possesses both the diffusion and advection terms of the N.–S. equations, it embodies all the main mathematical features of the N.–S. equations if the nonhomogeneous function in the equation is nonzero. On the other hand, it is
well know that direct numerical simulations (DNS) of N.-S. equations at high-Reynolds number, $Re$, are impossible at the present time or in the near future due to the restricted computer resources. The 1-D character of Burgers’ equation allows us to sidestep the difficulties encountered. Therefore, many numerical studies on turbulent flows have been conducted by solving Burgers’ equation. Successful examples can be found in McDonough and Bywater [1], Love [2], and more recently in Adams and Stolz [3]. In [1] the effects of large scale motion on the small scale motion of turbulence were revealed, while in [2] several subgrid-scale models for large-eddy simulation (LES) were tested successfully. The work in [3] employed a subgrid-scale deconvolution approach to simulate the isothermal shock problem governed by inviscid and viscous Burgers’ equations.

Despite the fact that numerous investigations have been performed on the solutions of Burgers’ equation, only three types of solutions have been employed in previous studies. The first type involves the shock problem. The solution of Burgers’ equation exhibits one-dimensional shock wave features (see, for example, Jeng and Meecham [4], [3], Khosla and Rubin [5]). It should be emphasized, however, that shock solutions are only one simple type of Burgers’ equation solution; there are more complicated solutions if the nonhomogeneous function is not zero. The second type includes solutions such as sine functions (see, for example, Basdevant et al. [6] and Dakhoul and Bedford [7]) which change either in time or space, or both. This type of solution is driven by rather simple nonhomogeneous functions (or initial conditions if the nonhomogeneous function is zero) constructed in the obvious way once the desired solution behavior has been selected. The third type is random solutions. These solutions exhibit random features (so called Burgers’ turbulence) if the nonhomogeneous function changes randomly in time or the initial condition changes randomly in space with a zero nonhomogeneous function. The statistical properties of Burgers’ turbulence have been investigated by many authors (see,
for example, Qian [8], [1], [2], Keleti and Reed [9] and Mizushima and Saito [10]). Among other things, it has been shown that such Burgers’ turbulence can exhibit the $-5/3$ Kolmogorov inertial range exponent [8] .

From the above discussions, it is noticed that while the first two types of solutions of Burgers’ equation have little, if any thing, to do with turbulence, the third type of solutions has some features of turbulence. However, from the theory of dynamical systems we do not expect solutions to the N.-S. equations to be random unless these equations are randomly forced. Furthermore, since the objective of most numerical simulations of Burgers’ equation is to study the performance of new numerical algorithms for use in N.-S. algorithms, the random feature of the solution makes completely valid assessment difficult. In the present paper, a new 1-D Burgers’ equation turbulent solution will be presented. It will be shown that this model not only possesses turbulent features, but also has the deterministic characteristics of a N.-S. solution. We will compare the proposed model with experimental turbulent velocity time series to show that the deterministic solution of the Burgers’ equation can exhibit the very complicated behavior of real turbulent flow. It is proposed that models of this type could provide a useful tool for validation of numerical algorithms intended for use in turbulence simulations.

During the past decade, use of high-order numerical algorithms in the context of finite-difference methods has become relatively popular in computational fluid dynamics. In particular, high-order methods have been used in turbulence simulations, for example, by Ghosal [11] and Kravchenko and Moin [12]. However, based on the present exact solution of Burgers’ equation, we will show that the high-order truncation error of typical discrete methods may become very large and thus may not be neglected. This implies that attempting to employ high-order methods for solving equations exhibiting erratic, turbulent-like solutions may fail to produce expected results.
The remainder of this paper is organized as follows. In Sec. 2 we present details of the proposed Burgers’ equation model problem, and in Sec. 3 we highlight the similarities of these solutions to real turbulent flows. In Sec. 4 we investigate smoothness of the 1-D Burgers’ equation turbulence model presented here as understanding of this is crucial to the assessment of performance of any applied numerical procedure. In the final section we summarize the work and present concluding remarks.

2. A BURGERS’ EQUATION MODEL PROBLEM

In this section we first review the form of Burgers’ equation used in the present work, and then we describe a forcing function designed to produce complicated turbulent-like spatio-temporal behaviors.

The form of Burgers’ equation considered here is

$$u_t + \frac{1}{2}(u^2)_x - \frac{1}{Re} u_{xx} = -P_x, \quad x \in (a, b), \quad t \in (t_0, t_f],$$

(1)

with $u(x, t)$ the time- and space-dependent variable analogous to velocity in the N.–S. equations. Subscripts denote partial differentiation except in the right-hand side term $P_x$. This is a nonhomogeneous function that plays the role of a pressure gradient (hence, the suggestive notation); but here it must be prescribed and is in no way related to a divergence-free condition as is the case for the N.–S. equations. Similar to the fact that the pressure field strongly influences the flow field of a N.–S. flow, this nonhomogeneous function has a powerful effect on the form of solutions to Eq. (1). Indeed, as should be obvious, and as we will demonstrate herein, any desired solution nature can be chosen, and a corresponding forcing function can then be constructed to produce this. In particular, neither the “N-wave” nor the related “Burgulence” are necessarily the outcome of a
Burgers’ equation simulation.

There are two objectives for solving Burgers’ equations employing new numerical algorithms in the context of turbulence simulation. The first is to compare numerical results obtained by this method with the exact solutions of the original governing equation and thus validate the technique. This requires that exact solutions of Eq. (1) be easily obtained either analytically or numerically. The second objective is to apply the new numerical algorithm employed for solving Burgers’ equation to solve turbulent flow problems. Therefore, it is important that the solutions of Eq. (1) exhibit the erratic behavior of N.-S. turbulent solutions in general.

Thus, we have designed a forcing function that is completely deterministic, but exhibiting irregular (but not actually chaotic) behavior in both space and time. This is given by

\[
P_x(x, t) = 105\pi \sum_{j=1}^{M} \left\{ \frac{A_j^{2-\alpha} \sin^{19} f_j}{Re} \left[ 20 \cos^2 f_j - \sin^2 f_j \right] - A_j^{1-\alpha} \sin^{20} f_j \cos f_j \right\} \\
- \frac{525\pi}{A_j^{2\alpha-1}} \left( \sum_{j=1}^{M} \sin^{20} f_j \cos f_j \right) \left( \sum_{j=1}^{M} \sin^{21} f_j \right),
\]

(2)

where \(A_j\) and \(f_j\) are defined as follows:

\[
A_j = e^{j^{0.31}}, \quad f_j = A_j\pi (x + 0.8 + t), \quad j = 1, \ldots, M,
\]

(3)

and \(\alpha\) in Eq. (2) is a prescribed constant for each specific nonhomogeneous function. Once \(P_x\) is assigned, Eq. (1) will have a unique solution under prescribed boundary and initial conditions given as

\[
u(x, t) = 5 \sum_{j=1}^{M} \frac{1}{A_j^\alpha} \sin^{21} f_j.
\]

(4)

This is obtained by assigning initial and boundary conditions consistent with this function
at boundary points, and at the initial time. For large $M$, Eq. (4) can exhibit very complicated quasiperiodic oscillations with respect to both space and time, but it is nonetheless in $C^\infty$. If $\alpha$ is chosen appropriately, the oscillations are similar to the turbulent fluctuations of real fluid flows (recall the Hopf–Landau theory of turbulence, Landau and Lifshitz [13]).

3. MODEL COMPARISON

In this section we compare the exact Burgers’ equation solution proposed in the above section with experimental turbulent velocity time series to demonstrate that its complicated spatio-temporal behavior is closer to actual physical turbulence than the typical “Burgulence” that is sometimes studied. This is important for testing numerical methods to be implemented for solving the N.-S. equations at high $Re$ for obvious reasons.

Figure 1 compares the oscillations of Eq. (4) for $\alpha = 1$ and $M = 500$ with a physical experimental time series. The experimental velocity is measured by 1-D hot wire anemometry (HWA) for flow over a turbulator inside a channel with flow Reynolds number of $10^5$ (see Rocalawski et al. [14]). Since only the absolute value of the velocity component can be obtained by 1-D HWA, absolute values of $u$ in Eq. (4) at a certain point of space ($x = 0.7$) are shown in Fig. 1(a). It should be noted that temporal behavior of the solution at other points is similarly erratic. If we look at the spatial distribution of the solution at any specific instant, $u$ also changes in an erratic turbulent-like manner. In addition, to facilitate comparison, the variables shown in both Figs. 1(a) and 1(b) are normalized from 0 to 1. It can be seen that much of the experimental flow structure shown in Fig. 1(b) is reproduced in Fig. 1(a) in at least a qualitative sense. In particular, one can identify at least three main types of temporal “structures” (in the sense discussed in McDonough et al. [15] and Yang et al. [16]) in both parts of the figure, namely, high-amplitude rela-
tively widely-spaced spikes, low-amplitude, high-frequency oscillations, and oscillations of somewhat intermediate amplitude and frequency. While the distributions of these are not the same, the two figures are qualitatively similar. In addition, it is important to observe that N-wave formation, commonly viewed as a hallmark of Burgers’ equation solutions, is not apparent in the computations.

![Figure 1](image.png)

Figure 1: Comparison of Eq. (4) with experimental velocity time series; (a) computational, (b) experimental.

For further comparison, Fig. 2 illustrates power spectra of the time series shown in Fig. 1. Again, the power spectra shown in both Figs. 2(a) and 2(b) are normalized from 0 to 1. The computed power spectrum has somewhat higher power at low frequencies, and the experimental one exhibits essentially no decay at high frequencies, with evidence of “noisy quasiperiodicity” described by McDonough and Huang [17] in the context of a related discrete dynamical system. The computational one must be of this type, by construction,
although this is somewhat less evident in the figure. Thus, despite some specific differences in details, the general qualitative features are quite similar, demonstrating that Eq. (4) can exhibit physically-realistic behavior analogous to turbulent flows.

Figure 2: Comparison of power spectrum of Eq. (4) with that of experimental velocity time series; (a) computational, (b) experimental.

4. MODEL REGULARITY

In this section we investigate the smoothness of the exact solution of Burgers’ equation. We will study the magnitudes of derivatives calculated using the analytical formula Eq. (4) to evaluate the truncation error of Taylor expansions of the erratic function. Results of this will indicate that use of high-order methods with their attendant very-high order derivative dominant errors may be inappropriate for solutions exhibiting the erratic or chaotic behaviors of high-\(Re\) (and evidently turbulent) N.–S. solutions.
It should be pointed out that for the turbulence problem, the magnitude of high-order derivatives may increase rapidly as the order of the derivative goes up; i.e., solutions are not in $C^\infty$. The model problem described above also exhibits a trend of increasing derivatives (but only through a finite order). This is indicated in Fig. 3 which displays the $L^2$-norm of derivatives of various orders through eighth. These results are obtained by analytically differentiating Eq. (4), averaging in time, and then calculating

$$\| \frac{\partial^n u}{\partial x^n} \| = \left( \int_a^b \left| \frac{\partial^n u}{\partial x^n} \right|^2 \ dx \right)^{1/2},$$

without use of any numerical approximations. It is observed that the $L^2$-norm of the eighth-order derivative (equivalently, the $H^8$-norm of $u$) reaches $\sim 10^{33}$ while the $L^2$-norm of the first derivative only approaches $\sim 10^3$. This suggests a possible difficulty with high-order finite-difference methods.

We explore this further in Fig. 4 which displays the spatial distributions of fourth and eighth derivatives of the exact solution of Burgers’ equation, Eq. (4), but now multiplied by a selected grid spacing $h$, further demonstrating this growth and indicating that if a Taylor expansion converges at all, it may do so only very slowly. Moreover, this shows that the dominant truncation error of a typical sixth-order method would be on the order
of $10^4$ greater than that of a second-order method for a solution of this nature. As in Fig. 3, the derivatives in Fig. 4 are calculated from analytical formulas. The spatial step size $h$ indicated in the figure is chosen to be $1/511$, a typical underresolved step size employed in numerical simulations reported elsewhere [18]. From Fig. 4 it is observed that the plot corresponding to the eighth-order derivative has a generally much larger magnitude than does that for the fourth-order derivative despite its significantly smaller coefficient. This shows that high-order terms in a truncation error expansion cannot be neglected in this case, and a finite-difference approximation will fail.

But the phenomenon shown in Fig. 4 occurs only for excessively large $h$ (i.e., in the presence of underresolution); if $h$ is sufficiently small the high-order terms will decrease as the order increases, consistent with usual expectations. To demonstrate this, we show in Fig. 5 the same two high-order terms as shown in Fig. 4, but with $h = 1/8191$. It is interesting to note that in the context of finite-difference or finite-volume numerical
Figure 5: Spatial distributions of two high-order derivatives of Eq. (4), \((h = 1/8191)\); (a) fourth derivative, (b) eighth derivative.

Simulations this spatial grid size is the maximum value permitted to successfully conduct DNS of the Burgers’ equation problem considered here. We also checked the magnitude of the same two high-order terms with \(h = 1/4095\) and found that the magnitude of the eighth-order derivative is still larger than that of the fourth-order one, suggesting possible failure of the Taylor expansion of the truncation error to converge. We comment that use of high-order methods with their attendant very-high order derivative dominant errors will be inappropriate for solutions exhibiting the qualitative features of Eq. (4), and hence, also for high-\(Re\) turbulent solutions.

5. CONCLUDING REMARKS

In this paper we presented a 1-D Burgers’ equation turbulence model. The model can generate complicated spatio-temporal behaviors that are similar to those of actual physical turbulence. Because the model is deterministic and moreover possesses an analytical
solution, it exhibits significant potential for testing numerical algorithms in the context of turbulent simulations.

Analysis of the high-order derivatives of the exact solution of Burgers’ equation corresponding to this model indicates that high-order terms in a truncation error expansion cannot be neglected, and a finite-difference approximation will fail unless grid spacing is quite small.

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