RADIATIVE TRANSFER IN THREE-DIMENSIONAL
RECTANGULAR ENCLOSURES CONTAINING
INHOMOGENEOUS, ANISOTROPICALLY
SCATTERING MEDIA

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Abstract—Radiative transfer in a three-dimensional rectangular enclosure containing radiatively
participating gases and particles is studied using the first- and third-order spherical harmonics
approximations. Inhomogeneities in the radiative properties of the medium, as well as in the
radiation characteristics of the boundaries, are allowed for. The scattering phase function is
represented by the delta-Eddington approximation, and it is assumed to be a function of the
location in order to account for density variation of the particles in the medium. Numerical
solutions of the model equations are obtained using a finite difference scheme. For the purpose of
validating the P_2-approximation, the results are compared with those based on Hottei's zonal
method.

1. INTRODUCTION

Interest in radiative heat transfer has increased greatly during the last 25 years, mainly
because of wide applications to such systems as combustion chambers, furnaces, glass
and crystal processing, fibrous and porous insulations, and the atmosphere. General and
accurate solutions of the radiative transfer equation (RTE) are required, especially for
multidimensional geometries. These needs have caused the development of many methods,
but there is still a lack of accurate, easy-to-apply, two- and three-dimensional models,
which are compatible with finite difference schemes for determining flow fields and which
also allow for inhomogeneities in the medium as well as on the boundaries.

In coal-fired furnaces, for example, most of the heat is transferred by radiation. In
such systems, assumptions of an isothermal medium and isothermal boundaries are not
realistic. In addition, the combustion gases (H_2O, CO_2, CO), as well as particles such as
pulverized coal, char, fly-ash or soot, and their volumetric averages are also functions of
location. Some recent studies have shown that the effects of particles in coal-fired furnaces
would be as high as 30% of total radiative heat transfer. These particles emit, absorb and
scatter radiation. Corresponding phase functions are usually highly forward peaked, and
therefore isotropic scattering or nonscattering approximations are questionable. The phase
function depends on the particle size; hence, it may vary from one location in a furnace
to another according to the spatial distribution of the particles. The radiation properties
of combustion gases are temperature dependent and consequently, radiative transfer in
the system depends also very strongly on location in the furnace. Therefore, the species
and the temperature distributions in the combustion products should be considered
simultaneously in obtaining the contribution of the gases to the radiative heat flux. Thus,
for accurate heat-transfer predictions, the radiative transfer model should allow for spatial
distributions of temperature and concentrations of combustion gases and particulate
species. Also, the variation of surface emittances and reflectances of the furnace walls
should be properly modeled.

An extensive survey of the latest multidimensional radiative transfer models has been
prepared. The most desirable model is the one which would solve the general form of
the radiative transfer equation in two- and three-dimensional enclosures exactly. This is
very difficult because of the long-distance nature of radiation. Some exact formulations
for absorbing, emitting and scattering media have recently been reported. An exact
solution of RTE for any geometry can be also obtained by employing stochastic models
such as the Monte Carlo method. On the other hand, the zonal method is the most
widely used model for engineering radiative heat transfer calculations in multidimensional enclosures. Unfortunately, it cannot readily be adapted when scattering particles are present in the medium. Neither of these models are compatible with the finite difference methods for determining the flow and temperature fields. Usually approximations such as the moment, spherical harmonics \((P_N)\), discrete ordinates \((S_N)\), or some hybrid formulations are preferable to calculate radiative flux in multidimensional enclosures when the flow field must also be considered since they are compatible with finite-difference schemes. Radiation transfer in furnaces is usually anisotropic and higher order approximations must be employed to obtain realistic solutions, especially when particles are present. Recently, it has been shown that the fourth-order discrete ordinates \((S_4)\) and the third-order spherical harmonics \((P_3)\) approximations yield much better predictions for radiative transfer than the lower order ones \((S_2\) and \(P_1)\) in two-dimensional geometries. The higher order approximations \((S_4\) and \(P_3)\) result only in minimal improvement in accuracy but cause a great increase in the analytical and the computational effort.

The spherical harmonics approximation is as elegant as it is tedious. It can be easily used with finite difference schemes required for flow-field calculations and can yield quite accurate radiative heat flux predictions, especially for optical thicknesses greater than unity.\(^{11,12}\) It has also been stated that, for general multidimensional radiative transfer calculations, the \(P_3\)-approximation may be considered to be the optimal choice as far as accuracy and computational effort are concerned.\(^{14,16}\)

The \(P_3\)-approximation has been formulated for two-dimensional rectangular enclosures containing an absorbing, emitting, and isotropically scattering homogeneous medium subject to diffusely emitting boundaries.\(^{12}\) However, for three-dimensional enclosures only some implicit expressions for general equations have been reported for the limiting case of nonscattering medium without giving complete boundary relations.\(^{11,13,14}\)

Recently, the formulation of the spherical harmonics \(P_3\)- and \(P_3\)-approximations for general rectangular, three-dimensional enclosures containing an absorbing, emitting, and anisotropically scattering, inhomogeneous, nonisothermal medium has been completed.\(^{19}\) The nonisothermal boundaries of the enclosure are assumed to be diffusely emitting and both diffusely and specularly reflecting and may be subjected to external incident diffuse and/or collimated fluxes from the surroundings. In this paper, we adapted the analysis for predicting radiative transfer in three-dimensional enclosures which may simulate furnaces. Our main objective is to present a methodology to solve the RTE in a combustion system containing radiating gases and particles but not to report detailed results.

2. ANALYSIS

2.1 Formulation

For an absorbing, emitting and scattering gas-particle mixture in LTE, the time-independent, monochromatic radiative transfer equation for three-dimensional rectangular enclosures (see Fig. 1) is\(^{15}\)

\[
\left[ \frac{1}{\kappa(x, y, z)} \left( \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \mu \frac{\partial}{\partial z} \right) + 1 \right] I(x, y, z, \theta, \phi) = J(x, y, z) + \frac{\omega(x, y, z)}{4\pi} \int_0^{2\pi} \int_0^{\pi/2} I(x, y, z, \theta', \phi') \Phi(\theta, \phi; \theta', \phi') \sin \theta' \, d\theta' \, d\phi',
\]

where \(\xi, \eta,\) and \(\mu\) are direction cosines such that

\[
\xi = \sin \theta \cos \phi, \quad \eta = \sin \theta \sin \phi, \quad \mu = \cos \theta,
\]

and \(J\) is the source function given as

\[
J(x, y, z) = (1 - \omega(x, y, z)) I_b[T(x, y, z)] + \frac{\omega(x, y, z)}{4\pi} \left\{ \int_{C_{x}} I_c(x_c, y_c) \Phi(\hat{\Omega}_c) e^{-\beta(x-x_c)} \right\} + \int_{\Delta \Omega} I_b(x_d, y_d) \Phi(\hat{\Omega}_d) e^{-\beta(x-x_d)} d\Omega_d.
\]
In this equation, \( I_b \) is Planck’s blackbody function, \( I_D \) and \( I_C \) are the incident diffuse and collimated beams on top surface of the enclosure at \((x_C, y_C)\) and \((x_D, y_D)\), respectively. Here, neither the temperature of the medium nor the radiative properties such as the extinction coefficient \( \beta(x, y, z) \), the single scattering albedo \( \omega(x, y, z) \), or the scattering phase function \( \Phi \) are assumed to be uniform; they may depend on position. The index of refraction of the medium is taken as being very close to unity.

The radiative transfer equation, Eq. (1), is difficult to solve analytically. The spherical harmonics approximation is now introduced, and the intensity as well as the scattering phase function are expressed by a series of spherical harmonics\(^{20}\)

\[
I(x, y, z, \vartheta, \phi) = \sum_{n=0}^{N} \sum_{m=-n}^{n} A_n^m(x, y, z) Y_n^m(\vartheta, \phi), \tag{4}
\]

or

\[
\Phi(\theta, \phi; \theta', \phi') = \sum_{n=0}^{N} \sum_{m=-n}^{n} a_n Y_n^m(\theta, \phi) Y_n^m(\theta', \phi'), \tag{5a}
\]

\[
\Phi(\theta, \phi; \theta', \phi') = \sum_{n=0}^{N} \frac{2n+1}{4\pi} a_n P_n(\cos \Psi), \tag{5b}
\]

where \( Y_n^m(\theta, \phi) \) represents the spherical harmonics, which are related to the associated Legendre functions, \( P_n^m(\cos \Psi) \), by

\[
Y_n^m(\theta, \phi) = (-1)^{\lfloor n/2 \rfloor} \frac{2n+1}{4\pi} \frac{(n-|m|)!}{(n+|m|)!} P_n^{|m|}(\cos \Psi) e^{im\phi}, \tag{6}
\]

here, \( \Psi \) is the scattering angle. The complex conjugates of the spherical harmonics are defined as

\[
Y_n^m(\theta, \phi) = (-1)^m Y_n^{-m}(\theta, \phi). \tag{7}
\]

They obey the following orthogonality condition

\[
\int_{4\pi} Y_n^m(\Omega) Y_n^m(\Omega') \, d\Omega = \delta_n \delta_{m'}, \tag{8}
\]

where \( \delta \) is the Kronecker delta function. The associated Legendre functions may be expressed in terms of Legendre polynomials by using the identity

\[
P_n^m(\mu) = (1 - \mu^2)^{m/2} \frac{d^m}{d\mu^m} P_n(\mu). \tag{9}
\]
In Eqs. (4) and (5), the upper limit $N$ for the index $n$ is known as the order of approximation. Exact solution of the RTE is obtained as $N \rightarrow \infty$. However, for practical purposes, $N$ should be chosen as a finite number, and usually odd numbers are assigned to it.\textsuperscript{21} By choosing $N$ as 1 or 3, $P_1$- or $P_3$-approximations are obtained, respectively. With increasing $N$, the accuracy as well as the complexity of the method increase. It is reported that for a nonscattering medium with uniform radiative properties in two- and three-dimensional enclosures the gain in accuracy of the $P_3$-approximation over the $P_3$-approximation is minimal, yet there is a dramatic increase in the programming and computation time requirements.\textsuperscript{14} For highly forward scattering in a planar medium, the accuracies of $P_3$- and $P_3$-approximations are comparable in predicting the reflectance and transmittance of a slab.\textsuperscript{22} Consequently, $N = 3$ appears to be a reasonable compromise for multidimensional geometries. In this study, only $P_1$- and $P_3$-approximations are considered.

In the spherical harmonics method, the moments of intensity are used as dependent variables in evaluating the intensity or radiative flux distributions. Transformation of the moment equations is carried out by using

$$
[\text{Eq. (1)}] \quad Y_{n}^{m} (\theta, \phi) \sin \theta \, d\theta \, d\phi
$$

for $n = 0, 1, \ldots, N$ and $m = -n, -n + 1, \ldots, n$. A close look at the definition of $Y_{n}^{m}$ reveals that it can be replaced with appropriate direction cosine products to perform the integrations required by Eq. (10). The moments are then obtained by integrating the intensity over all angles, after first multiplying by the appropriate direction cosines. Therefore, they are not direction dependent and are defined as

$$
I_0(x, y, z) = I(x, y, z, \theta, \phi) \sin \theta \, d\theta \, d\phi,
$$

$$
I_i(x, y, z) = \int_0^{2\pi} \int_0^{\pi} I_i(x, y, z, \theta, \phi) \sin \theta \, d\theta \, d\phi,
$$

$$
I_{ij}(x, y, z) = \int_0^{2\pi} \int_0^{\pi} I_{ij}(x, y, z, \theta, \phi) \sin \theta \, d\theta \, d\phi,
$$

$$
I_{ijk}(x, y, z) = \int_0^{2\pi} \int_0^{\pi} I_{ijk}(x, y, z, \theta, \phi) \sin \theta \, d\theta \, d\phi,
$$

with the "closure conditions," such as

$$
I_{i0} = -(3/35)I_0 + (6/7)I_{ii}, \quad I_{ij} = (3/7)I_{ij},
$$

$$
I_{ijk} = (4/35)I_0 - (1/7)I_{jk}, \quad I_{iij} = (1/7)I_{jij}.
$$

Here $i$, $j$, $k$ are direction cosines and each of these is either $\xi$, $\eta$ or $\mu$ [see Eq. (2)].

The radiative intensity distribution can be written explicitly only if the coefficients $A_n^m(x, y, z)$ in Eq. (4) are known. These are obtained in terms of the moments of the intensity by using the orthogonality relation of spherical harmonics and become zero when $n > N$ for $P_N$-approximation. They are presented in the Appendix. When these coefficients are known, the intensity distribution for the $P_3$-approximation becomes

$$
I(x, y, z, \xi, \eta, \mu) = \frac{1}{4\pi} \left\{ M_0 + M_1\xi + M_2\eta + M_3\mu + M_4\xi^2 + M_5\eta^2 + M_6\mu^2 \\
+ M_7\xi\eta + M_8\xi\mu + M_9\eta\mu + M_{10}\xi^3 + M_{11}\eta^3 \\
+ M_{12}\mu^3 + M_{13}\xi\eta\mu + M_{14}\xi\eta^2\mu \\
+ M_{15}\eta^2\mu + M_{16}\xi\mu^2 + M_{17}\eta\mu^2 \right\},
$$
Radiative transfer in three-dimensional rectangular enclosures

\[ M_0 = [(9/4)I_0 - (15/4)I_{33}], \quad M_1 = [(-15/2)I_1 + (105/2)I_{122}], \]
\[ M_2 = [(-15/2)I_2 + (105/2)I_{112}], \quad M_3 = [(75/4)I_3 - (105/4)I_{333}], \]
\[ M_4 = [(15/4)(I_{11} - I_{22})], \quad M_5 = -M_4, \]
\[ M_6 = [(45/4)I_{33} - (15/4)I_0], \quad M_7 = [15I_{12}], \]
\[ M_8 = [15I_{13}], \quad M_9 = [15I_{23}], \]
\[ M_{10} = [(35/2)(I_{111} - 3I_{122})], \quad M_{11} = [(35/2)(I_{222} - 3I_{112})], \]
\[ M_{12} = [(175/4)I_{333} - (105/4)I_3], \quad M_{13} = [105I_{123}], \]
\[ M_{14} = [(105/4)(I_{113} - I_{223})], \quad M_{15} = -M_{14}, \]
\[ M_{16} = [(105/2)(I_{133} - I_{122})], \quad M_{17} = [(105/2)(I_{233} - I_{112})]. \] (13)

For the \( P_1 \)-approximation the intensity distribution is simply

\[ I(x, y, z, \xi, \eta, \mu) = \frac{1}{4\pi} \{ I_0 + 3(\xi I_1 + \eta I_2 + \mu I_3) \}. \] (14)

By applying Eq. (10), the moments of the source function \( J \) can also be evaluated. Then, by switching \( I \)'s to \( J \)'s in Eqs. (11), the necessary \( J \)-moments are written implicitly. After lengthy and tedious algebraic manipulations, they can be expressed as

\[ \begin{align*}
J_0(x, y, z) &= 4\pi(1 - \omega_0)I_0[\mathcal{T}(x, y, z)] + I_D E_0 + I_C F_0, \\
J_j(x, y, z) &= I_D E_j + I_C F_j, \\
J_i(x, y, z) &= \delta_{ij} \frac{\pi}{4}(1 - \omega_0)I_0[\mathcal{T}(x, y, z)] + I_D E_i + I_C F_i, \\
J_{ijk}(x, y, z) &= I_D E_{ijk} + I_C F_{ijk}.
\end{align*} \]

where \( i, j, k \) are related to corresponding direction cosines and may take any value between 1 and 3. If one surface is subject to an incident diffuse flux, a point in the medium receives the corresponding diffuse radiation restricted by the solid angle between that point and the surface, and the \( E \) and \( F \)-functions of Eqs. (15) are to specify these solid angles at any location in the medium (see Fig. 2). For a uniform diffuse flux incident on the boundary at \( z = 0 \), they are given as

\[ \begin{align*}
E_m = & \frac{\omega}{4\pi} \int_0^1 L_m \int_0^{\pi} \int_0^{2\pi} \Phi(\hat{\Omega}; \hat{\Omega}_D)e^{-\beta z/\mu} d\mu d\phi d\Omega = E_m^I + E_m^{II} + E_m^{III} + E_m^{IV}, \\
F_m = & \frac{\omega}{4\pi} \int_0^1 L_m \Phi(\hat{\Omega}; \hat{\Omega}_C)e^{-\beta z/\mu} d\Omega \\
\end{align*} \]

with

\[ \begin{align*}
m = 0, & \quad L_m = 1, \\
m = i, & \quad L_m = l_i, \\
m = ij, & \quad L_m = l_i l_j, \\
m = ijk, & \quad L_m = l_i l_j l_k.
\end{align*} \] (17)

and

\[ \begin{align*}
E_m^I &= \frac{\omega}{4\pi} \int_0^1 \int_{\phi_i}^{\phi_j} \int_{\mu_{min}}^{\mu_{max}} \Phi(\hat{\Omega}; \hat{\Omega}_D)e^{-\beta z/\mu} d\mu d\phi d\Omega, \\
E_m^{II} &= \frac{\omega}{4\pi} \int_0^1 \int_{\phi_i}^{\phi_j} \int_{\mu_{min}}^{\mu_{max}} \Phi(\hat{\Omega}; \hat{\Omega}_C)e^{-\beta z/\mu} d\mu d\phi d\Omega.
\end{align*} \] (18a, 18b)
Fig. 2. The critical $\phi$ and $\theta$ angles are shown for any point $(x, y, z)$ in the medium. There are diffuse and collimated fluxes incident on the surface at $z = 0$.

\[ E^{\text{III}}_m = \frac{\omega}{4\pi} \int_{\Omega} \int_{\phi_1}^{\phi_2} \int_{\phi_3}^{\phi_4} \Phi(\Omega; \Omega_D) e^{-\beta z / \mu} \, d\mu \, d\phi \, d\Omega, \]  
\[ E^{\text{IV}}_m = \frac{\omega}{4\pi} \int_{\Omega} \int_{\phi_1}^{\phi_2} \int_{\phi_3}^{\phi_4} \Phi(\Omega; \Omega_D) e^{-\beta z / \mu} \, d\mu \, d\phi \, d\Omega, \]

where $\phi_1$, $\phi_2$, $\phi_3$, $\phi_4$ and $\mu_{\text{min}}$ are so-called critical angles, and they can be expressed in terms of spatial coordinates (see Fig. 2), such as

\[ \phi_1 = +\arccos \left[ \frac{|x_0 - x|}{[(x_0 - x)^2 + (y_0 - y)^2]^{1/2}} \right], \]  
\[ \phi_2 = -\arccos \left[ \frac{|x_0 + x|}{[(x_0 + x)^2 + (y_0 - y)^2]^{1/2}} \right] + \pi, \]  
\[ \phi_3 = +\arccos \left[ \frac{|x_0 + x|}{[(x_0 + x)^2 + (y_0 + y)^2]^{1/2}} \right] + \pi, \]  
\[ \phi_4 = -\arccos \left[ \frac{|x_0 - x|}{[(x_0 - x)^2 + (y_0 + y)^2]^{1/2}} \right] + 2\pi, \]

\[ \mu_{\text{min}} = \begin{cases} 
\cos \left[ \arctan \left( \frac{x_0 - x}{z \cos \phi} \right) \right], & \phi_4 < \phi < \phi_1, \\
\cos \left[ \arctan \left( \frac{y_0 - y}{z \sin \phi} \right) \right], & \phi_1 < \phi < \phi_2, \\
\cos \left[ \arctan \left( \frac{-x_0 - x}{z \cos \phi} \right) \right], & \phi_2 < \phi < \phi_3, \\
\cos \left[ \arctan \left( \frac{-y_0 - y}{z \sin \phi} \right) \right], & \phi_3 < \phi < \phi_4.
\end{cases} \]
These relations are given for a uniform diffuse flux, $I_D$, incident on only one surface (i.e. $z = 0$ surface). If the assumption that $I_C$ and $I_D$ are uniform can not be justified, then $E$-integrals given by Eqs. (16)-(18) are to be evaluated by including the $I_D(x, y, z = 0)$ function under the integral signs. On the other hand, if more than one surface is subjected to external incident diffuse fluxes, then the resulting expressions are easily written by using a similar analysis or superposition relations. Since the primary application here is to combustion chambers and furnaces, there is no need to consider external incident fluxes on the boundaries. In the remaining part of the analysis we assume that the incident radiation field terms, $I_C$ and $I_D$, are identically zero, and the $J$-function consists only of the Planck’s blackbody function.

2.2 Phase function approximation

The scattering phase function is given by Eq. (5) and is quite difficult to handle, especially for three-dimensional geometries. The resulting expressions may be simplified by using an approximation for the phase function. The $\delta$-Eddington phase function approximation\textsuperscript{23} appears to be a reasonable one, since it accounts for highly forward scattering by the particles in the medium and may be written as

$$\Phi(\theta, \phi; \theta', \phi') = 2f \delta(1 - \cos \Psi) + (1 - f)(1 + 3g \cos \Psi), \quad (21a)$$

where $\Psi$ is the scattering angle such that

$$\cos \Psi = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi'). \quad (21b)$$

In terms of direction cosines, this equation becomes

$$\cos \Psi = \xi \xi' + \eta \eta' + \mu \mu'. \quad (21c)$$

Here the parameters $f$ and $g$ are related to the coefficients of series expansion of the phase function and are defined as

$$g = (a_1 - a_2)/(1 - a_2), \quad (22a)$$
$$f = a_2, \quad (22b)$$

where

$$(2n + 1)a_n = (1/4\pi) \int_0^{2\pi} \int_0^\pi \Phi(\theta, \phi; \theta', \phi') P_n(\cos \Psi) \sin \theta \, d\theta \, d\phi. \quad (22c)$$

2.3 Governing equations

The intensity distributions were given in Section 2.1 in terms of the moments of the intensity. To express the intensity or the radiative flux, these moments must be evaluated. In this section, the radiative transfer equation is transformed into partial differential equations in terms of the moments. First, it is rewritten and normalized after substituting the phase function approximation [Eqs. (21)] and the moments of intensity [Eqs. (11)] into Eq. (1), i.e.

$$\left[ \frac{\partial}{\tau_0 \partial \bar{x}} + \frac{\partial}{\tau_0 \partial \bar{y}} + \mu \frac{\partial}{\tau_0 \partial \bar{z}} + 1 \right]J(x, y, z, \theta, \phi)$$
$$= (1 - \omega_0)I_0[T(x, y, z)] + \frac{\omega_0}{4\pi} [I_0 + 3g(I_1 + \eta I_2 + \mu I_3)], \quad (23)$$

where

$$\bar{x} = x/z_0, \quad \bar{y} = y/z_0, \quad \bar{z} = z/z_0 \quad (24a)$$
are normalized $\bar{x}$-, $\bar{y}$- and $\bar{z}$-coordinates, and

$$\tau_0 = \beta(1 - \omega f)z_0, \quad \omega_0 = (1 - f)\omega/(1 - f \omega)$$

(24b)

are the normalized optical thickness in the $z$ direction and the normalized single scattering albedo, respectively.

To obtain the governing equations, the normalized form of the RTE [Eq. (23)] is substituted into Eq. (10). This procedure yields a single partial differential equation in terms of the zeroth moment of intensity, $I_0$ (or so-called irradiance) for the $P_1$-approximation, viz.

$$\nabla^2 I_0 = (\Gamma_{ii} + \Gamma_{ij} + \Gamma_{kk})I_0 = A(I_0 - 4\pi I_0[T(x, \bar{y}, \bar{z})]),$$

(25)

where

$$\Gamma_{ij} = \partial^2/\partial x_i \partial x_j$$

(26)

and the indices $i$ or $j$ (or $k$) take any value between 1 and 3, and 1 corresponds to the $\bar{x}$-axis, 2 to the $\bar{y}$- and 3 to the $\bar{z}$-axes. In Eq. (25),

$$A = 3(1 - \omega_0)(1 - \omega_0 \omega)\tau_0^2.$$  

(27)

For the $P_3$-approximation, use of the operations defined by Eq. (11) results in a series of partial differential equations in terms of zeroth, first-, second- and third-order moments of intensity. Following a lengthy and tedious procedure, these equations can be reduced to six linear partial differential equations in terms of second-order moments, $I_{11}, I_{22}, I_{33}, I_{12}, I_{13}$ and $I_{23}$. In shorthand notation, these equations are written as

$$[B\Gamma_{ii} - 4(\Gamma_{ij} + \Gamma_{kk})]I_{ii} + [3\Gamma_{ii} + CT_{jj} + \Gamma_{kk}]I_{jj} + [3\Gamma_{ii} + \Gamma_{ij} + CT_{kk}]I_{kk} + [D\Gamma_{ij}]I_{ij}$$

$$+ [D\Gamma_{ik}]I_{ik} + [D\Gamma_{jk}]I_{jk} = \begin{bmatrix} -\nu_0 I_{ii} + \frac{\omega_0}{3}(I_{jj} + I_{kk}) + \frac{4\pi}{3}\tau_0^2(1 - \omega_0)I_0[T] \\ 35\end{bmatrix},$$

(28)

$$[F\Gamma_{ij}][I_{ii} + I_{jj}] + [2\Gamma_{ij}]I_{kk} + [G(\Gamma_{ii} + \Gamma_{jj}) - 5\Gamma_{kk}]I_{ij}$$

$$+ [H\Gamma_{jk}]I_{ik} + [H\Gamma_{jk}]I_{jk} = -35I_{ij},$$

(29)

where the operators $\Gamma$ are defined by Eq. (26). Note again that the indices $i$, $j$ and $k$ of the moments take any value between 1 and 3, and consequently each of these equations correspond to three partial differential equations. If there are external diffuse and/or collimated fluxes incident on the boundaries some additional terms appear at the right-hand side of Eq. (28) and Eq. (29). The coefficients in these equations are

$$B = -(27 + 21\alpha), \quad C = -(4 + 7\alpha), \quad D = -(30 + 28\alpha),$$

$$E = -(10 + 14\alpha), \quad F = -(8 + 7\alpha), \quad G = -(15 + 7\alpha),$$

$$H = -(10 + 7\alpha), \quad \alpha = \omega_0 g/(1 - \omega_0 g), \quad \nu_0 = (1 - \omega_0/3).$$

(30)

All of these coefficients are in functional form and may vary from one location to another in the medium because the normalized albedo $\omega_0$, the optical thickness $\tau_0$, or the phase function parameters $f$ and $g$ are not assumed constant. Simultaneous solution of these six differential equations gives the second-order moments of intensity, and from these all other moments can be obtained by tracing the procedure back.

The divergence of total radiative flux (integrated over all wavelengths) in the medium is defined by

$$\nabla \cdot q' = S(\bar{x}, \bar{y}, \bar{z}),$$

(31)
where $S$ is the inhomogeneous heat source [in kW/m$^3$]. This can be related to the blackbody function and the zeroth moment of intensity such that

$$[4\pi I_0[T(\bar{x}, \bar{y}, \bar{z})] - I_0] = z_0 S(\bar{x}, \bar{y}, \bar{z})/\tau_0(1 - \omega_0).$$

Note that when the medium is in radiative equilibrium, i.e. $S = 0$, the temperature distribution in the medium is obtained from Eq. (32) by simply equating the blackbody function to the irradiance $I_0$. If a heat source in the medium is given rather than a temperature distribution, then the blackbody function in the governing equations is to be written in terms of the irradiance $I_0$ and the source $S$ by using Eq. (32).

The radiative flux distribution $q_i$, in the $\bar{x}$-, $\bar{y}$- or $\bar{z}$-directions can be related to the first moment of intensity by

$$q_i(\bar{x}, \bar{y}, \bar{z}) = I_i(\bar{x}, \bar{y}, \bar{z}).$$

Also, $q_i$ can be divided into the forward and backward components such that

$$q_i^+(\bar{x}, \bar{y}, \bar{z}) = \pm \frac{1}{4}\{\frac{3}{2}I_0 \pm 2I_i \pm \frac{3}{2}I_0\},$$

where $i = 1$ for $\bar{x}$-, 2 for $\bar{y}$- and 3 for $\bar{z}$-directions. The radiative flux distribution can be calculated easily from either Eq. (33) or (34) once the appropriate moments of intensity have been determined.

2.4 Boundary conditions

There are two different types of boundary conditions to be applied for the spherical harmonics approximation, namely, Mark's and Marshak's.\textsuperscript{21,24} For higher order approximations, the Mark's boundary conditions are preferable;\textsuperscript{21} however, for $P_1$- or $P_2$-approximations Marshak's boundary condition yields more accurate results.\textsuperscript{18,21} The reason for this is that for the lower-order approximations, the average intensity leaving the boundaries can represent the intensity distribution more realistically. The Marshak's boundary conditions are obtained by taking the integral of the intensity over the appropriate hemispheres such that

$$\int_{2\pi} J(r_{wall}, \hat{\Omega}) Y_{2n-1}(\hat{\Omega}) d\Omega = \int_{2\pi} h_w(r_{wall}, \hat{\Omega}) Y_{2n-1}(\hat{\Omega}) d\Omega,$$

where $n = 1, 2, \ldots, (N + 1)/2$, and $-2n + 1 \leq m \leq 2n + 1$. In this equation, $\hat{\Omega}$ is for the solid angle, and $h_w$ corresponds to the intensity at a diffusely emitting, and specularly and diffusely reflecting, opaque boundary and is given as

$$h_w(r_w, \theta, \phi) = \epsilon_w I_0[T_w] + \rho_w I'(r_w, \theta', \phi') + \frac{\rho_w d}{\pi} \int_{2\pi} J(r_w, \theta', \phi') d\theta' d\phi',$$

where the superscript $i$ denotes the incident intensity and $I_j$ corresponds to the normal of the boundary surface for which $h_w$ is written. For each surface, the limits on $\theta$ and $\phi$ are different and are to be determined accordingly. Table 1 summarizes the limits used in the analysis.

In performing the integrations defined by Eq. (35), the spherical harmonics $Y_n^{m-1}$ can be replaced by the multiples of direction cosines

$$Y_n^{m}(\theta, \phi) = l_i,$$

$$Y_n^{m}(\theta, \phi) = l_i l_j l_k,$$
Table 1. The $\theta$ and $\phi$ limits on each boundary.

<table>
<thead>
<tr>
<th>Boundary</th>
<th>$\theta$</th>
<th>$\phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x=0$</td>
<td>(0) - ($\pi$)</td>
<td>($\frac{\pi}{2}$) - ($\frac{\pi}{2}$)</td>
</tr>
<tr>
<td>$x=x_0$</td>
<td>(0) - ($\pi$)</td>
<td>($\frac{\pi}{2}$) - ($\frac{3\pi}{2}$)</td>
</tr>
<tr>
<td>$y=0$</td>
<td>(0) - ($\pi$)</td>
<td>($\pi$) - ($2\pi$)</td>
</tr>
<tr>
<td>$y=y_0$</td>
<td>(0) - ($\pi$)</td>
<td>(0) - ($2\pi$)</td>
</tr>
<tr>
<td>$z=0$</td>
<td>(0) - ($\frac{\pi}{2}$)</td>
<td>(0) - ($\pi$)</td>
</tr>
<tr>
<td>$z=z_0$</td>
<td>($\frac{\pi}{2}$) - ($\pi$)</td>
<td>(0) - ($\pi$)</td>
</tr>
</tbody>
</table>

where $i, j$ or $k$ may take any value between 1 and 3. Equation (37a) is to be used in obtaining the boundary conditions for the $P_1$-approximation and Eq. (37b) for the $P_3$-approximation.

In practical combustion chambers, the walls can usually be assumed to be diffusely reflecting. Therefore, the specular component of reflectivity $\rho_s'$ can be neglected. Then, by using Eqs. (11), (12), (36) and (37) in Eq. (35), the boundary conditions for the $i$th surface are obtained in terms of moments of the intensity

$$3I_0 + 15I_{ii} = -16(1 + 2\lambda_w)I_i = 32\pi I_b[T_w],$$

$$-(2 + 5\lambda_w)I_0 + 15(2 + \lambda_w)I_{ii} = -32(1 + \lambda_w)I_{ii} = 32\pi I_b[T_w],$$

$$8 + 5\lambda_w)I_0 = -(16)I_{ii} + (64)(1 + \lambda_w)I_{ij}$$

$$+ 10(1 + \lambda_w)(I_{ij} - I_{kk}) - 15\lambda_w I_{ii} = 32\pi I_b[T_w],$$

$$5I_{ij} = -(8)I_{ij} = 0,$$

$$5I_{jk} = -(16)I_{jk} = 0,$$

where $\pm$ corresponds to the surfaces at the positive or negative directions. Here $\lambda_w$ represents the ratio of diffuse reflectivity to emissivity. Subscript $w$ denotes the wall properties which may be functions of location and may be different for each wall.

The boundary conditions for the $P_1$-approximation on the $i$th surface are

$$I_0 \pm \frac{2}{3} (1 + \alpha)(1 + 2\lambda_w) \frac{\partial I_0}{\partial x_i} = 4\pi I_b[T_w],$$

where $\alpha$ is given by Eq. (30). For the $P_3$-approximation boundary conditions, Eqs. (39)-(42) are to be used in conjunction with the governing partial differential equations Eqs. (28) and (29), which are in terms of second-order moments of the intensity. Accordingly, these boundary conditions are also to be given in terms of second-order moments, which can be accomplished by using the following relations:

$$I_i = -(1 + \alpha)\left(\frac{\partial I_{ij}}{\partial x_j}\right),$$

$$I_{ii} = \frac{3}{35} \frac{\partial I_0}{\partial x_i} - \frac{3}{7} \frac{\partial I_{ii}}{\partial x_i} - \left[\frac{3}{7} + \frac{3}{5} \alpha\right] \left(\frac{\partial I_{ij}}{\partial x_j}\right),$$

$$I_{ij} = \frac{1}{35} \frac{\partial I_0}{\partial x_i} - \left[\frac{1}{7} + \frac{1}{5} \alpha\right] \left(\frac{\partial I_{ij}}{\partial x_j}\right) - \frac{1}{7} \left(\frac{\partial I_{ij}}{\partial x_i} + 2 \frac{\partial I_{ij}}{\partial x_j}\right),$$

$$I_{jk} = -\frac{1}{7} \left(\frac{\partial I_{jk}}{\partial x_i}\right).$$
Radiative transfer in three-dimensional rectangular enclosures

Table 2. Boundary equations for the second-order moments of the intensity.

<table>
<thead>
<tr>
<th>Moments of Intensity</th>
<th>( x = 0 ) or ( x_0 )</th>
<th>( y = 0 ) or ( y_0 )</th>
<th>( z = 0 ) or ( z_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Moments</td>
<td>Eq.</td>
<td>Moments</td>
</tr>
<tr>
<td>( I_{11} )</td>
<td>( \xi^3 )</td>
<td>(39)</td>
<td>( \eta )</td>
</tr>
<tr>
<td>( I_{12} )</td>
<td>( \xi \eta )</td>
<td>(40)</td>
<td>( \eta )</td>
</tr>
<tr>
<td>( I_{13} )</td>
<td>( \xi \mu )</td>
<td>(40)</td>
<td>( \eta \mu )</td>
</tr>
<tr>
<td>( I_{22} )</td>
<td>( \xi^2 )</td>
<td>(41)</td>
<td>( \xi^2 )</td>
</tr>
<tr>
<td>( I_{23} )</td>
<td>( \xi \mu )</td>
<td>(41)</td>
<td>( \xi \mu )</td>
</tr>
<tr>
<td>( I_{33} )</td>
<td>( \eta^2 )</td>
<td>(41)</td>
<td>( \eta^2 )</td>
</tr>
</tbody>
</table>

All of these relations are for the \( i \)th surface, and \( i \) may be 1 (\( =\bar{x} \)), 2 (\( =\bar{y} \)), or 3 (\( =\bar{z} \)). The symbol \( (\cdot)_j \) denotes summation over \( j \). Note that these boundary relations include some additional terms if there are external diffuse and/or collimated fluxes incident on one of the surfaces. The list of the boundary equations to be used with appropriate moments is summarized in Table 2.

2.5 Method of solution

Equation (25) for the \( P_1 \)-approximation or Eqs. (28) and (29) for the \( P_3 \)-approximation are elliptic, linear partial differential equations. As long as all the radiative properties are in functional form and the boundary conditions are not vacuum type, these equations can be solved numerically. In this study, a general computer code ELLPACK, which is a proprietary general elliptic partial differential equation solver developed at Purdue University, is used to obtain the numerical solutions of the equations.

The results for the \( P_1 \)-approximation can be readily obtained by employing the ELLPACK code. These results are used as an initial guess for the \( P_3 \)-approximation, and six partial differential equations [Eqs. (28) and (29)] are solved simultaneously for the second order moments of intensity. Following this, an iterative scheme is employed and the calculated numerical results for either temperature or heat fluxes are compared with results from the previous iteration. When a predetermined convergence criterion is satisfied, the temperature as well as heat flux distributions in the medium are calculated using the converged second-order moments of intensity.

3. RESULTS AND DISCUSSION

Analytical solutions of the \( P_1 \)-approximation are possible for some limited conditions, and they are desirable to evaluate the accuracy of the general numerical scheme. Such a solution was obtained by assuming that all radiative properties of the medium are homogeneous, and the temperature of at least five surfaces forming the enclosure are identical. To satisfy the convergence criterion of 1 K for the temperature, 30 eigenvalues of the Sturm–Liouville system were employed. From comparisons of the temperatures and the radiative fluxes it was clear that the difference between the predictions of the exact analytical and numerical methods was at most 5%. Use of a finer grid increased the accuracy of the numerical results only slightly.

In obtaining the results for the \( P_3 \)-approximation numerically, first the \( P_1 \)-approximation results are used to initialize the moments of intensity. Then the governing equations are solved iteratively. Usually five to seven iterations are enough to obtain convergence for the temperature as well as radiative flux distributions in the medium.

Various grid schemes were tested, and it is found that when the number of grid points is increased the number of iterations required decreases. The results obtained using a finer grid differed relatively little from those obtained using a coarser grid. This difference
became smaller as the number of grid points increased. Most of the results reported in the paper were obtained with a $7 \times 7 \times 11$ nonuniform grid scheme.

Some representative results for the $P_3$-approximation are presented in a way of an example of modeling an idealized furnace. The configuration of the system under consideration is shown in Fig. 1. For the base calculations, the following data are used:

Medium:

$$\beta = 0.5 \text{ m}^{-1}, \quad \omega = 0, \quad S = 5.0 \text{ kW/m}^3,$$

$$x_0 = 2 \text{ m}, \quad y_0 = 2 \text{ m}, \quad z_0 = 4 \text{ m}.$$

Boundaries:

$$z = 0, \quad T = 1200 \text{ K}, \quad \epsilon = 0.85,$$

$$z = z_0, \quad T = 400 \text{ K}, \quad \epsilon = 0.70,$$

$$\text{others}, \quad T = 900 \text{ K}, \quad \epsilon = 0.70.$$

When other values are employed in sample calculations, they are indicated either directly on the figures or in the captions.

In Fig. 3, the temperature distributions at three axial locations of the furnace are compared with the predictions based on Hottel's zonal method.\(^\dagger\) At the center of the enclosure, the $P_3$-approximation yields very accurate temperature predictions. Near the hot $(\bar{z} = 0.10)$ or the cold $(\bar{z} = 0.90)$ walls, the results deviate from those obtained using Hottel's method by as much as 5%; however, with increasing optical thickness, this deviation decreases.

The heat fluxes at the cold and hot surfaces are compared in Fig. 4 with those obtained from the zonal analysis for the different extinction coefficients. For $\beta = 0.5 \text{ m}^{-1}$, the $P_3$-approximation predicts the surface heat fluxes by as much as 20% higher than the zonal method. However, with increasing optical thickness this difference decreases to under 10%. The biggest discrepancy between results based on the two methods occurs near the corners. At the interior nodes the agreement is usually good. From the nature of the differential approximation, it can be claimed that spherical harmonics $P_3$-approximation is accurate for large optical thicknesses. Indeed, for optical thicknesses less

\(^\dagger\) A three-dimensional computer program developed by T.-H. Song of the School of Mechanical Engineering, Purdue University.
than unity, it has been shown that the differential approximations are not very reliable.\textsuperscript{11,12} Note that the heat flux at both the hot and the cold walls decreases with increasing optical thickness. This is an expected behavior since with increasing extinction coefficient more energy is trapped in the medium.

Emissivity of the bounding walls is a very important parameter in calculating the radiative flux distribution in an enclosure. In Fig. 5, the heat fluxes at the hot and cold surfaces are compared for three different wall emissivities. As expected, decreasing emissivity (or increasing reflectivity) results in a more uniform heat flux distribution in the medium. It is worth noting that a change in wall emissivity from $\epsilon = 0.6$ to 0.95 may cause about 50–60\% difference in the heat flux. Therefore, in order to obtain realistic
predictions of radiative transfer in the medium the wall emissivities are to be specified with great care.

The effect of scattering on the radiative flux distributions at the hot and cold walls is illustrated in Fig. 6. It is clear that with increasing scattering, the radiative flux at the hot surface decreases dramatically, and for $\omega = 0.7$ the hot surface receives heat from the medium. On the other hand, at the cold surface, the radiative flux for the highly scattering medium ($\omega = 0.7$) is about three times larger than for the non-scattering medium ($\omega = 0.0$). Even for a moderate single scattering albedo ($\omega = 0.3$), the radiative fluxes at the hot as well as the cold surfaces may be as much as 50% different from those for a non-scattering medium. The results show that if scattering by the particles, such as pulverized coal, char or fly-ash in the medium is not accounted for, then the analysis may yield unrealistic results.

![Fig. 6. Effect of single scattering albedo on the local heat fluxes at the hot and cold walls.](image)

![Fig. 7. The three-dimensional temperature distribution in the medium, $\beta = 2m^{-1}$ and $\omega = 0.2$.](image)
There are several key points highlighted in the document:

1. **Radiative Transfer in Three-Dimensional Rectangular Enclosures**
   - The three-dimensional temperature distribution in the medium is shown in Fig. 7. Note that the temperature near the hot surface is higher than the hot surface temperature, which is a result of the heat source as well as high extinction coefficient ($\beta = 2 \text{ m}^{-1}$) and scattering ($\omega = 0.2$) in the medium.

2. **Effect of Scattering Phase Functions**
   - The effect of phase functions (or, say, the type of particles in the medium) on radiative fluxes at the hot and the cold surfaces are compared in Fig. 8. The parameters $f$ and $g$ of the $\delta$-Eddington phase function approximation are obtained from the phase functions reported. For highly forward scattering particles the parameter $f$ is significant. From normalizations given in Eq. (24b) it is clear that the higher the $f$-parameter, the smaller is the value of the normalized albedo. From the comparisons given in Fig. 6, it was concluded that when the effect of scattering decreases, a more uniform radiative flux distribution results in the medium. Therefore, it is reasonable to expect a smoother flux distribution in the medium when the $f$-parameter of the phase function approximation becomes higher. Indeed, this type of behavior is clear from Fig. 8. When the particles are highly forward scattering (second and third phase functions), the radiative flux distribution in the medium tends to be more uniform than those for nearly or completely isotropic scattering particles. The type of particles in the medium influence the radiative flux distribution.

3. **Divergence of Radiative Fluxes**
   - In this study only the temperature and the heat flux distributions at the walls are reported. However, in most engineering applications, the divergence of radiative flux vector in the medium is required [see Eq. (31)] for coupling the RTE with the energy equation. It is worth noting that the divergence of the radiative flux is usually smoother than the flux itself, and if accurate flux distributions are obtained, then reliable radiative flux divergence distribution results should be assured. On the other hand, the equations given in this study are in general form, and can be reduced to one- and two-dimensional Cartesian coordinate system equations (such as reported in Refs. 11 and 12) after some manipulations.

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**Fig. 8.** Effect of scattering phase functions (PF-0: $f = g = 0.0$; PF-I: $f = 0.111, g = 0.215$; PF-II: $f = 0.639, g = 0.773$; PF-III: $f = 0.781, g = 0.868$) on the local heat fluxes at the hot and cold walls: $\epsilon = 0.8$ for all walls, $\beta = 1 \text{ m}^{-1}$ and $\omega = 0.5$. 

---
A typical run for the computer program (five to seven iterations) required about 35-40 min on the Vax 11-780 machine. However, on the Cyber-205 digital computer of the Purdue University Computing Center, the same run took only about 90 sec. With some optimization it is possible to reduce this time at least 30%.

4. CONCLUSIONS

A three-dimensional radiative transfer model based on the spherical harmonics approximation has been developed. The resulting governing equations for the $P_1$- or $P_3$-approximations are solved by using the accurate ELLPACK code. The model allows for inhomogeneities in the medium and at the boundaries. Therefore, variations in gas particle concentrations, heat sources and scattering properties can be accounted for. The computer time requirements show that the model can be successfully used for accurate radiative flux calculations in furnaces since it is compatible with the finite-difference equations for the flow field.

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REFERENCES

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APPENDIX: $A_n^m$ COEFFICIENTS OF INTENSITY DISTRIBUTION

The $A_n^m$ coefficients of radiative intensity distribution [Eq. (3)] can be evaluated by using the orthogonality relation of Legendre polynomials. Implicitly, they are written as either

$$A_n^m(\tau) = \int_{\Delta \omega} (-1)^{n+1} P_n^m(\hat{n}) \tilde{I}(\hat{n}) \sin \phi \, d\phi$$

or

$$A_n^m(\tau) = \frac{2n+1}{4\pi} (n+1)^{1/2} \int_{\Delta \phi} \tilde{I}(\hat{n}) P_n^m(\cos \theta) e^{-im\phi} \, d\phi.$$

These expressions can be used to evaluate the $A_n^m$ coefficients for a given intensity distribution function $\tilde{I}(\hat{n})$.
Radiative transfer in three-dimensional rectangular enclosures

For the $P_2$-approximation, we need 15 $A_n^m$ coefficients. They are

\[
A_0^0 = (1/4\pi)^{1/2}[I_0],
\]

\[
A_1^0 = (3/4\pi)^{1/2}[I_1],
\]

\[
A_{1\pm1} = \pm(3/8\pi)^{1/2}[I_1 \pm iJ_1],
\]

\[
A_2^0 = -(5/16\pi)^{1/2}[3I_2 - I_6],
\]

\[
A_{2\pm1} = \pm(15/8\pi)^{1/2}[I_2 \pm iJ_2],
\]

\[
A_{2\pm2} = (15/32\pi)^{1/2}[I_{11} - I_2 \pm (-2)iJ_{12}],
\]

\[
A_3^0 = (7/16\pi)^{1/2}[5I_{22} - 3I_5],
\]

\[
A_{3\pm1} = \pm(21/64\pi)^{1/2}[-(5I_{111} - I_3) \pm i(5I_{223} - I_3)],
\]

\[
A_{3\pm2} = (105/32\pi)^{1/2}[I_{113} - I_{223} \pm (-2)iJ_{123}],
\]

\[
A_{3\pm3} = \pm(35/64\pi)^{1/2}[3I_1 - 4I_{111} - 3I_{111} \pm i(3I_2 - 4I_{222} - 3I_{222})],
\]

(A3)

where $i$ is the unit imaginary number $(-1)^{1/2}$. 