1.a) Determine the type number of the following open-loop Z-domain transfer functions:

i) \( G(z) = \frac{10(z + 1)^2}{z(z - 1)} \)

ii) \( G(z) = \frac{10(z + 1)^3}{z^2(z - 1)} \)

iii) \( G(z) = \frac{10(z + 1)^3}{z(z - 1)^2} \)

**Solution:** i) type 1  ii) type 1 iii) type 2

b) Let \( T_s = 100 \) msec and find the static error coefficients \( K_p, K_v \) and \( K_a \) for problem 1a) (assume unity feedback).

**Solution:**

i) \( K_p = \lim_{z \to 1} G(z) = \infty \), \( K_v = \lim_{z \to 1} \frac{z-1}{T_s} G(z) = 40 / T_s = 400 \), \( K_a = \lim_{z \to 1} \left( \frac{z-1}{T_s} \right)^2 G(z) = 0 \)

ii) \( K_p = \lim_{z \to 1} G(z) = \infty \), \( K_v = \lim_{z \to 1} \frac{z-1}{T_s} G(z) = 80 / T_s = 800 \), \( K_a = \lim_{z \to 1} \left( \frac{z-1}{T_s} \right)^2 G(z) = 0 \)

iii) \( K_p = \lim_{z \to 1} G(z) = \infty \), \( K_v = \lim_{z \to 1} \frac{z-1}{T_s} G(z) = \infty \), \( K_a = \lim_{z \to 1} \left( \frac{z-1}{T_s} \right)^2 G(z) = 80 / (T_s)^2 = 8000 \)

c) For each of the closed-loop unity feedback systems in part a), find:

i) \( e_s \) due to a step 
ii) \( e_s \) due to a ramp 
iii) \( e_s \) due to a parabola 

i) \( e_s \) due to a step = 1/(1+Kp) = 0 (part 1a)i)), = 0 (part 1a)ii)), = 0 (part 1a)iii))

ii) \( e_s \) due to a ramp = 1/Kv = 1/400 (part 1a)i)), = 1/800 (part 1a)ii)), = 0 (part 1a)iii))

iii) \( e_s \) due to a parabola = 1/Ka = \( \infty \) (part 1a)i)), = \( \infty \) (part 1a)ii)), = 1/8000 (part 1a)iii))

d) Sketch the root locus for the systems in part a).

**Solution:**

![Root locus of Prob. 1a) part i)](image)

![Root locus of Prob. 1a) part ii)](image)

![Root locus of Prob. 1a) part iii)](image)

2. Given the system

\[
\begin{align*}
\text{W}(s) & \quad \text{+} & \quad \text{T} & \quad \text{G}(z) & \quad \text{=} & \quad \text{G}_{\text{o}}(z) & \quad \text{=} & \quad \frac{10}{z(z + 8)} \\
\text{G}_{\text{na}}(z) & \quad \rightarrow & \quad \text{Y}(s)
\end{align*}
\]

a) Find a Z-domain model for the open-loop system including the ZOH if \( T_s = 10 \) msec.

**Solution:**

\[
G_{na}(z) = \left[ z^{-1} \right] \left[ G_{na}(s)/s \right] = \left[ z^{-1} \right] \left[ 10/(s(z + 8)) \right] = \left[ z^{-1} \right] 1.2019 \times 10^{-6} (z+1)^5((z-1)^7(z-0.9231)) = 1.2019 \times 10^{-6} (z+1)^5((z-1)(z-0.9231))
\]
b) What type number is your Z-domain model? What type is the original model?

**Solution:** Since \( G_{zo}G(z) \) has one open-loop pole at \( z=1 \), it is a type one system. Since the original \( G_{zo}G(s) \) has one open-loop pole at \( s=1 \), it is also a type one system.

c) Find \( e_u \) due to a step, \( e_u \) due to a ramp, and \( e_u \) due to a parabola for your uncompensated model.

**Solution:**

\[
K_p = \lim_{z \to 1} G_{zoh} G(z) = \infty, \quad K_v = \lim_{z \to 1} \frac{z-1}{z T_s} G(z) = 1.25 \times 10^{-4} / T_s = 1.25 \times 10^{-2}, \quad K_a = \lim_{z \to 1} \left( \frac{z-1}{z T_s} \right)^2 G(z) = 0
\]

Thus, \( e_{step} = 1/(1+K_p) = 0 \), \( e_{ramp} = 1/K_v = 80 \), \( e_u \) due to a parabola = \( 1/K_a \) = \( \infty \).

d) Given the following transient specifications: \( t_s < 0.4 \) sec and \( M_p < 2\% \). Illustrate the region of the s-plane and the z-plane where we must place our dominant poles to satisfy these specs.

**Solution:** From the first spec, we find that \( t_s < 0.4 \Rightarrow \zeta \omega_n > 10 \). From the second spec we find that \( M_p = \frac{-\zeta^2}{\sqrt{1-\zeta^2}} \times 100\% \). Or, solving this relationship for the damping coefficient we obtain, \( \zeta = \frac{\ln(m)^2}{\pi^2 + \ln(m)^2} \) where \( m = M_p/100\% = 0.02 \). Thus, \( \zeta = \frac{\ln(0.02)^2}{\pi^2 + \ln(0.02)^2} = 0.7797 = \cos \theta \). Hence, \( \theta = \cos^{-1}(\zeta) = \cos^{-1}(0.7797) = 38.77^\circ \).

Since we must have \( M_p < 2\% \), our constraint becomes \( \theta \leq 38.77^\circ \). In the s-plane, we obtain the following region:

In the z-plane, this region maps to:

e) Design \( G_c(z) \) (plus possibly a lag compensator) to meet the above specs plus the added spec that \( e_u \) due to ramp \( \leq 1/50 \).

**Solution:** We must first design a lead compensator to meet the transient specs. To be consistent, let’s pick the desired dominant poles to be \( s_1 = -11 + j6 \) which maps to \( z_1 = 0.8942 + j0.0537 \). Perhaps the uncompensated root locus will pass thru \( z1 \). To check, let’s sketch the uncompensated root locus of \( G_{zo}G(z) = 1.2019 \times 10^4 (z+1)^3(1/(z-1)/(z(z-0.9231))) \):
As can be seen, the root locus does not pass thru $z_1$. Next, let’s find the angle of deficiency (i.e., the amount of lead angle needed to bend the root locus thru $z_s$). The angle of deficiency is $\angle G_{lead}(z_1) = 180^\circ \times odd# - \angle G_{zoh}G(z_1)|_{z_1=0.89+j0.054} = 180^\circ \times odd# - 90.12^\circ = 89.88^\circ$. Therefore, our lead compensator must supply 89.88 degrees (this compares favorably to the solution in the s-plane which provided 89.77 degrees).

Let’s pick $-z_c$ to be as far to the left as possible while still having $-p_c$ to the right of the origin. Let $z_c = -0.8976$. Then $\angle G_{lead}(z_1) = 89.88^\circ = \angle(z_1 + 0.8976) - \angle(z_1 + p_c) = 92.63^\circ - \angle(z_1 + p_c)$ or $\angle(z_1 + p_c) = 351^\circ$. Solving for $p_c$: $IM(z_1) / RE(z_1 + p_c) = tan(213^\circ) or p_c = IM(z_1) / tan(351^\circ) - RE(z_1) = -0.0189$. The last step is to find $K_c$ from the magnitude condition: $K_c = 1 / |G_{zoh}G(z_1)(z_1 + z_c)/(z_1 + p_c)| = 12.910$. Thus, the lead compensator is $G_{lead}(s)=12.910(z-0.8976)/(z-0.0189)$. Finally, after performing the lead compensator design, we can find a lag compensator to put in series with $G_{lead}$. The desired $K_v$ is 50 to meet the $e_{ss}$ specs. Thus, $K_v = \lim_{z \to \infty} \frac{z}{T_s} = 16.84$. Thus, we need to increase the steady-state gain by a factor of $50/16.84=2.9694$ to meet our $e_{ss}$ specs. Therefore, Let $(1+z_{lag})/(1+p_{lag}) = 2.986$. Let’s pick $p_{lag} = -0.9999$. Thus, $z_{lag}=-0.9997$. To find $K_{lag}$, use the magnitude condition: $K_{lag} = 1 / |(z_{lag}/(z-0.9997))G_c G_{zoh}G(z)|_{z_1=0.89+j0.054} = 10015$. Therefore, the lag compensator which must be inserted in series with $G_{lead}(z)$ is $G_{lag}(z) = 1.0015(z-0.9997)/(z-0.9999)$.

f) Simulate a step and ramp response of your closed-loop compensated digital system using dlsim() in MATLAB. Measure $t_s, M_p, and e_{ss}$ (both step and ramp).

Solution: The closed-loop step and ramp responses are shown below:
g) How do your digital compensator numbers compare to the $G_c(z)$ we designed for this system in HWs #15 and #16?

**Solution:** The numbers are almost identical except for $K_{lead}$ which is about 100 times greater.