Solution to HW#22

1.a) Design a PD (ultimate lead compensator) to meet the transient specs given in problem 2 on HW21.

Sol’n: Desired dominant poles are at s=-4+j4 and s=-4-j4. The angle of deficiency is still the same as it was for the lead design: \[ \angle G_{pd}(s_1) = 180^\circ \times \text{odd#} - \angle G(s_1) = 180^\circ \times \text{odd#} - \angle \frac{20}{\text{Re}(s_i)+4\text{Im}(s_i)+40} = 180^\circ - 128.66^\circ = 51.34^\circ. \] Therefore, our PD compensator must supply 51.34 degrees. The form of \( G_{pd} \) is \( G_{pd}(s) = K(s+z). \) Then \[ \angle G_{pd}(s_1) = 51.34^\circ = \angle(s_1 + z). \] Solving for \( z \):

\[ IM(s_1)/RE(s_1 + z) = \tan(51.34^\circ) \text{or } z = IM(s_1)/\tan(51.34^\circ) - RE(s_1) = 7.2. \] The last step is to find \( K \) from the magnitude condition: 

\[ K = 1/|G(s_1)(s_1 + 7.2)| = 8. \] Thus, the PD compensator is \( G_{pd}(s) = 8(s+7.2) = 8s + 57.6. \)

b) Sketch the resulting compensated root locus of \( G(s)G_{pd}(s) \)

Sol’n: Lead Root Locus

c) Design a PI (ultimate lag compensator) compensator for \( G(s) = \frac{10}{s+4} \) such that the following specs are met:

1. \( t_s = 1 \) second
2. Damping coefficient of \( \zeta = 0.707 \)
3. \( \text{ess|step} = 0 \)

Sol’n: Notice we need a type 1 system to meet the \( e_{ss} \) specs. If we lump the needed integrator in with \( G(s) \) then our new open-loop transfer function is \( G(s)/s = \frac{10}{s(s+4)} \). Desired dominant poles are again at s=-4+j4 and s=-4-j4. By lumping the integrator in with \( G(s) \), we can now can do a PD compensator design on the new open-loop transfer function, \( G(s)/s \). The angle of deficiency is \[ \angle G_{pd}(s_1) = 180^\circ \times \text{odd#} - \angle G(s_1)/s_1 = 180^\circ \times \text{odd#} - \angle \frac{10}{s_1(s_1+4)} = 180^\circ - 225^\circ = 45^\circ. \] Therefore, our PD compensator must supply 45 degrees. The form of \( G_{pd} \) is \( G_{pd}(s) = K(s+z) \). Thus, \[ \angle G_{pd}(s_1) = 45^\circ = \angle(s_1 + z). \] Solving for \( z \):

\[ IM(s_1)/RE(s_1 + z) = \tan(45^\circ) \text{or } z = IM(s_1)/\tan(51.34^\circ) - RE(s_1) = 8. \] The last step is to find \( K \) from the magnitude condition: 

\[ K = 1/|G(s_1)(s_1 + 7.2)| = 0.4. \] Thus, the PI compensator is \( G_{pi}(s) = 0.4(s+8)/s = 0.4s + 3.2/s. \)

d) Sketch the resulting compensated root locus of \( G(s)G_{pi}(s) \)

Sol’n:
**e)** Design a PID (ultimate lead-lag compensator) compensator for \( G(s) = \frac{10}{s + 4} \) such that the following specs are met:

1. \( ts = 0.5 \) second
2. Damping coefficient of \( = 0.707 \)
3. \( ess\text{|ramp} = 0.1 \)

**Sol’n:**

A PID compensator is of the form of \( G_{PID}(s) = K_D s + K_P + K_I / s \). The PID design process is unique among our root locus-based compensator designs. The first step, is to satisfy the steady-state error specs. Note that the desired \( K_v \) is \( 10 = K_i \times 10/4 \) which implies that \( K_i = 4 \). Now, \( ess\text{|ramp} = 0.1 \) as specified. The next step is to find \( K_P \) and \( K_D \) from:

\[
K_P + K_D s_1 = -1 / G(s_1) - K_f / (s_1).
\]

Here, once we pick a pair of desired dominant poles, then \( K_P \) and \( K_D \) are uniquely determined. By virtue of the transient specifications, we have no option in finding \( s_1 \). In fact, \( s_1 = -8+j8 \) is the only possible location to produce a damping ratio of 0.707 and a settling time of 0.5. Thus, \( K_P + K_D s_1 = K_P - 8K_D + j8K_D = 0.65 - j0.55 \). By equating the imaginary part of this equation we find that \( K_D = 0.55/8 = -0.0688 \). By substituting this value and equating the real part of our equation, we find that \( K_P = 0.65-K_D \times Re(s_1) = 0.1 \). Thus, \( G_{PID}(s) = K_D s + K_P + K_I / s = -0.0688 s + 0.1 + 4/s \). According to MATLAB, the closed-loop poles for the PID compensated system are -8+j8 and -8-j8. The zeroes of the closed-loop system are located at 8.34 and -6.94.

**f)** Sketch the resulting compensated root locus of \( G_{PID}(s)G(s) \)

**Sol’n:**

Note that there are two disturbing facts about this design. The overall \( K=K_D \) is NEGATIVE and there is an open-loop (and closed-loop) zero in the RHP. Thus, to perform the root locus, we actually have to use the “complementary” root locus plot which is based upon \( K \) being negative: Here is a plot of both the root-locus and the complementary root locus for this problem. The regular root locus is shown in cyan (light blue) and the complementary root locus is shown in brown:
Don’t worry about the complementary root locus!

2. For my lead design in HW#21, I found the following closed-loop transfer function:

\[
\frac{GcG}{1+GcG} = \frac{15872(s + 6.9360)}{(s + 4 - j4)(s + 4 + j4)(s + 33.59)(s + 102.41)} = \frac{15872(s + 6.9360)}{(s^2 + 8s + 32)(s + 33.59)(s + 102.41)}
\]

If my system were truly a classic second order system with dominant poles at \(s_1 = -4+j4\) and \(s_2 = -4-j4\), the closed-loop transfer function would be:

\[
\frac{G_{\text{classic}}}{1+G_{\text{classic}}} = \frac{32}{s^2 + 8s + 32}
\]

I have gone back and plotted my lead compensated step response versus what a classic 2\(^{nd}\) order step response would look like for the (see plot below).

---

a) What are the theoretical values of \(t_p\), \(M_p\) , and \(t_e\) for dominant poles at \(s_1 = -4+j4\)?
Solution: \( tp = \frac{\pi}{\omega_d} = 0.7854, \ M_p = 4.32\% \), and \( ts = 1 \) sec.

b) Given your answer to part d) which plot is which (hint: the classic second-order system should have the correct overshoot and settling time)?

Solution: The green plot has the theoretical values of \( tp, M_p, \) and \( ts \).

c) What are the values of \( tp, M_p, \) and \( ts \) for my lead design from the Matlab step response?

Solution: Looking at the blue plot, \( tp = 0.6 \) sec, \( M_p = 8 \% \), and \( ts = 0.98 \) sec.

d) Note that the zero at \( s = -6.936 \) is fairly close to the dominant poles at \( s_1 = -4 + j4 \). Given the difference between the two plots, what is your conjecture as to the effect of a zero near a pair of dominant poles?

My conjecture is that the presence of a zero near our closed-loop dominant poles causes more overshoot, a faster peak time, and a slightly faster settling time. See the handout on the effect of poles-zeros for more illustrations of what effect extra poles and zeros might have on the classic step response.