# Diagonalizing and Whitening a Covariance Matrix

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3-25-03

Given two Gaussian r.v.s, $x_1$, $x_2$, with zero mean and covariance

$$C_{xx} = \begin{bmatrix} 1 & -3/4 \\ -3/4 & 1 \end{bmatrix} \tag{1}$$

We want to find a transformation

$$\underline{y} = A\,\underline{x} \tag{2}$$

where $\underline{y} = [\, y_1\, y_2]^T$, $\underline{x} = [\, x_1\, x_2]^T$ and $A$ is 2x2.

## Step 1: Find eigenvalues of $C_{xx}$.

The eigenvalues are found by setting the determinant of $C_{xx}$ - $\lambda_I\, \underline{I}$ equal to 0:

$$\left| C_{xx} - \lambda_i\, I \right| = 0 \tag{3}$$

For the values in Eq. (1) we have

$$\left(1-\lambda_i\right)^2 - 9/16 = 0 = \lambda_i^2 - 2\lambda_i + 7/16 \tag{4}$$

$$\lambda_1, \lambda_2 = 7/4, 1/4 \tag{5}$$

## Step 2: Find eigenvectors of $C_{xx}$.

We know the relationship of eigenvalues and eigenvectors is

$$C_{xx} e_i = \lambda_i e_i \tag{6}$$

and in matrix form

$$C_{xx} E = E\,\Lambda \tag{7}$$

where the columns of $E$ are the eigenvectors and the diagonal of $\Lambda$ contains the eigenvalues. Also note that for orthonormal case, $E^T = E^{-1}$

The eigenvalue/eigenvector relationship is

$$\left(C_{xx} - \lambda_i\, I\right) e_i = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_{11} - \lambda_i & c_{12} \\ c_{21} & c_{22} - \lambda_i \end{bmatrix} \begin{bmatrix} e_{1,i} \\ e_{2,i} \end{bmatrix} \tag{8}$$

The Eq. (8) will give two sets of equations but for reasons we will not go into, we will only use the first equation such that

$$\left(1-\lambda_i\right)e_{1,i} - \left(3/4\right)e_{2,i} = 0 \tag{9}$$

Let $i$ = 1 so Eq. (9) is

$$\left(1-7/4\right)e_{1,1} = \left(-3/4\right)e_{1,1} = \left(3/4\right)e_{2,1} \tag{10}$$

Let $i$ = 2 so Eq. (9) is

$$\left(1-1/4\right)e_{1,2} = \left(3/4\right)e_{1,2} = \left(3/4\right)e_{2,2} \tag{11}$$

We see that for one eigenvector $e_{1,1} = -e_{2,1}$ and for the other eigenvector $e_{1,2} = e_{2,2}$. The inner product of an eigenvector with itself is unity for an orthonormal eigenvector so if we let the element values have values

$$E = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{12}$$

## Step 3: Diagonalize $C_{yy}$

Letting $A = E^T$ such that

$$y_i = e_i^T \underline{x} = e_{1,i}x_1 + e_{2,i}x_2 \tag{13}$$

we can diagonalize $C_{yy}$ such that

$$C_{yy} = E\left\{\underline{y}\,\underline{y}^T\right\} = AE\left\{\underline{x}\,\underline{x}^T\right\}A^T = AC_{xx}A^T = E^T C_{xx} E = E^T E \Lambda = \Lambda \tag{14}$$

## Step 4: Whitening $C_{yy}$

From Eq. (14), we can see that if we weighted $A$ by $\Lambda^{-1/2}$ we would get an identity matrix for $C_{yy}$. The algebra for this would be based on letting $A = \Lambda^{-1/2}E^T$ such that

$$C_{yy} = E\left\{\underline{y}\,\underline{y}^T\right\} = AE\left\{\underline{x}\,\underline{x}^T\right\}A^T = AC_{xx}A^T = \Lambda^{-1/2}E^T C_{xx} E\,\Lambda^{-1/2} = \Lambda^{-1/2}E^T E \Lambda\Lambda^{-1/2} \tag{15}$$

and so because the eigenvalue matrix is diagonal we can use algebra to complete the whitening as

$$C_{yy} = \Lambda^{-1/2}\Lambda\Lambda^{-1/2} = \Lambda^{-1}\Lambda = I \tag{16}$$