LECTURE NOTES DAY 19
ECE EE640

Optimum Decision Boundaries
MAP, Neyman Pearson, Lagrange Multipliers 3-31-05

The MAP decision rule can also be put in the form of a maximum likelihood ratio (MLR)

$$\frac{P(H_1)\, f(y|H_1)}{P(H_0)\, f(y|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

$$\Rightarrow \quad \frac{f(y|H_1)}{f(y|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P(H_0)}{P(H_1)}$$

Cost Function

let C_{ij} be the cost associated with $P(D_i, H_j)$

Average cost is

$$\bar{C} = \sum_{i=0,1} \sum_{j=0,1} C_{ij} \, P(H_j) \, P(D_i | H_j)$$

$$= C_{00} \, P(D_0|H_0) P(H_0) + C_{10} \, P(D_1|H_0) P(H_0)$$

$$+ C_{01} \, P(D_0|H_1) \, P(H_1) + C_{11} \, P(D_1|H_1) P(H_1)$$

$$= C_{00} \, P(H_0) \int_{R_0} f_{y|H_0}(y|H_0) \, dy$$

$$+ C_{10} \, P(H_0) \int_{R_1} f_{y|H_0}(y|H_0) \, dy$$

$$+ C_{01} \, P(H_1) \int_{R_0} f_{y|H_1}(y|H_1) \, dy$$

$$+ C_{11} \, P(H_1) \int_{R_1} f_{y|H_1}(y|H_1) \, dy$$

Cost of making an incorrect decision may be higher than the cost of a correct one.

then $\quad C_{10} > C_{00} \quad$ and $\quad C_{01} > C_{11}$

Note $\quad \left.\begin{array}{l} R_1 \cup R_0 = U \\ R_1 \cap R_0 = \emptyset \end{array}\right\} \quad \int_{R_1} \sim dy = 1 - \int_{R_0} \sim dy$

$$\bar{C} = \underbrace{C_{10}\, P(H_0) + C_{11}\, P(H_{1})}_{} \Big\} \text{ constant}$$

$$+ \underbrace{\int_{R_0} \Big\{ P(H_1)\,(C_{01} - C_{11})\, f_{y|H_1}(y|H_1) - P(H_0)\,(C_{10} - C_{00}) \times f_{y|H_0}(y|H_0) \Big\}\, dy}_{\text{minimize}}$$

integrand is negative when

$$\underbrace{P(H_0)(C_{10} - C_{00})\, f_{y|H_0}(y|H_0) > P(H_1)(C_{01} - C_{11})\, f_{y|H_1}(y|H_1)}_{\text{choose } H_0}$$

$$\frac{P(H_1)(C_{01} - C_{11})\, f_{y|H_1}(y|H_1)}{P(H_0)(C_{10} - C_{00})\, f_{y|H_0}(y|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

or

$$L(y) = \frac{f_{y|H_1}(y|H_1)}{f_{y|H_0}(y|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P(H_0)}{P(H_1)} \frac{(C_{10} - C_{00})}{(C_{01} - C_{11})}$$

If $P(H_0)$ and $P(H_1)$ are not known then we have the "minmax" rule

~~For~~ Minimizes expected cost given $P(H_1)$ for which average cost is maximum

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Neyman- Pearson rule (N-P)

neither $P(H_0)$ or $P(H_1)$ is known

no cost assignments

Constrain $P(D_1 | H_0) < \gamma$

"false alarm"

$P(D_0 | H_1) \equiv$ "miss"

Minimize

The results for "minmax" and N-P is

$$L(y) = \frac{f_{y|H_1}(y|H_1)}{f_{y|H_0}(y|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$$

When ever you have a constraint and a minimization, think "Lagrange multiplier".

We can use Lagrange multiplier as a function

$$F = P_M + \lambda \left[P_F - \alpha' \right]$$

where we want to minimize P_M and constrain $P_F = \alpha'$

$$P_M = \int_{R_0} f_{y|H_1}(y|H_1)\, dy \;,\; P_F = \int_{R_1} f_{y|H_0}(y|H_0)\, dy$$

~~so~~

$$F = \lambda(1-\alpha') + \underbrace{\int_{R_0}}_{\int_{R_0} = \int_{-\alpha}^{\gamma}} f_{y|H_1}(y|H_1) - \lambda f_{y|H_0}(y|H_0)\,dy$$

$\lambda > 0$ will minimize

$\lambda < 0$ will maximize

Let $\frac{dF}{d\gamma} = 0$ where $\int_{R_0} = \int_{-\infty}^{\gamma}$

$$0 = \frac{dF}{d\lambda} = 1-\alpha' - \int_{R_0} f_{y|H_0}(y|H_0)\,dy$$

~~so~~ $f_{y|H_1}(\gamma|H_1) = \lambda f_{y|H_0}(\gamma|H_0)$

$$L(y) = \frac{f_{y|H_1}}{f_{y|H_0}} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

$$\alpha' = 1 - \int_{-\infty}^{\gamma} f_{y|H_0}(y|H_0)\,dy = \int_{\gamma}^{\infty} f_{y|H_0}(y|H_0)\,dy$$

find γ from $L(y)$

Reciever Operating Curve (ROC)