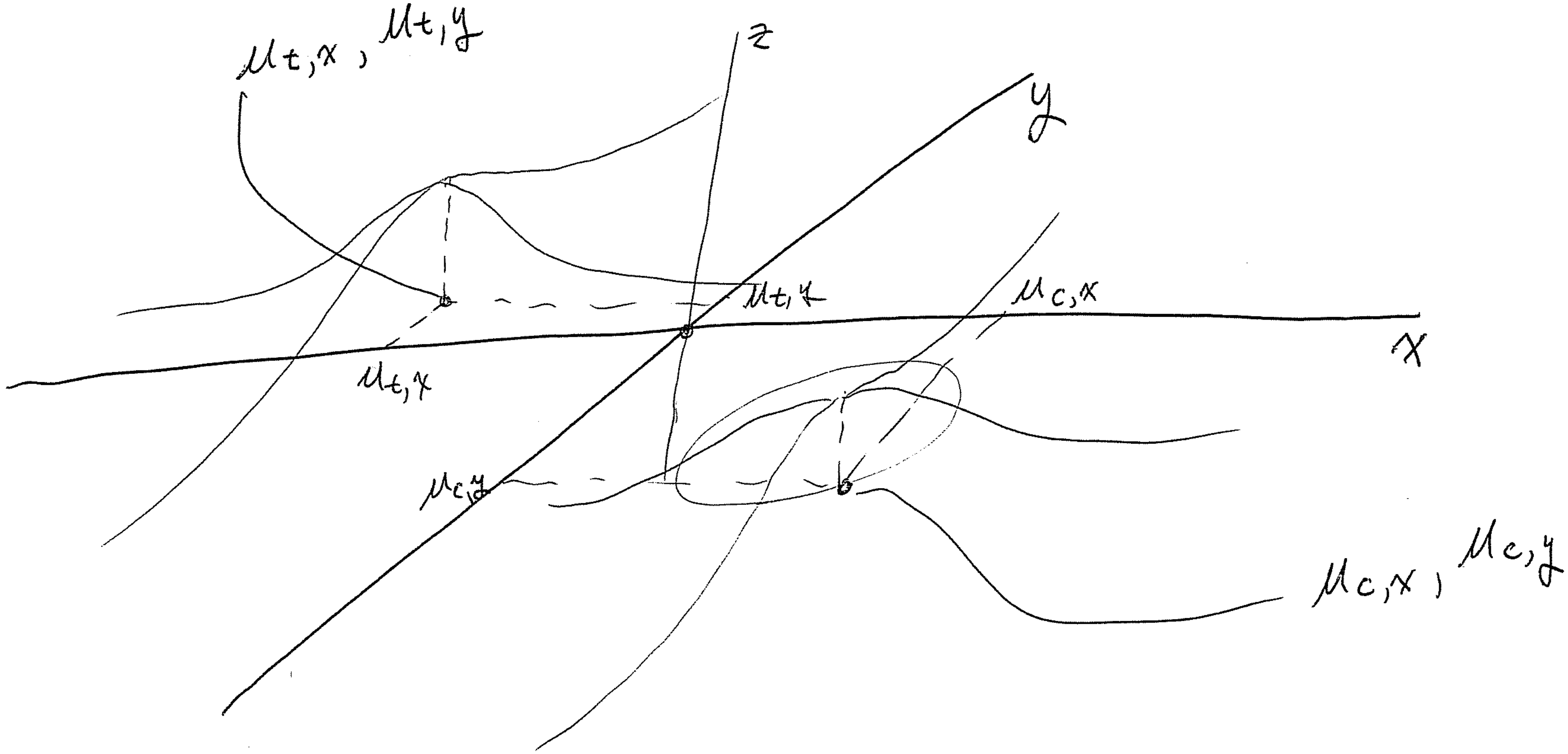
LECTURE NOTES DAY 18
ECE EE640

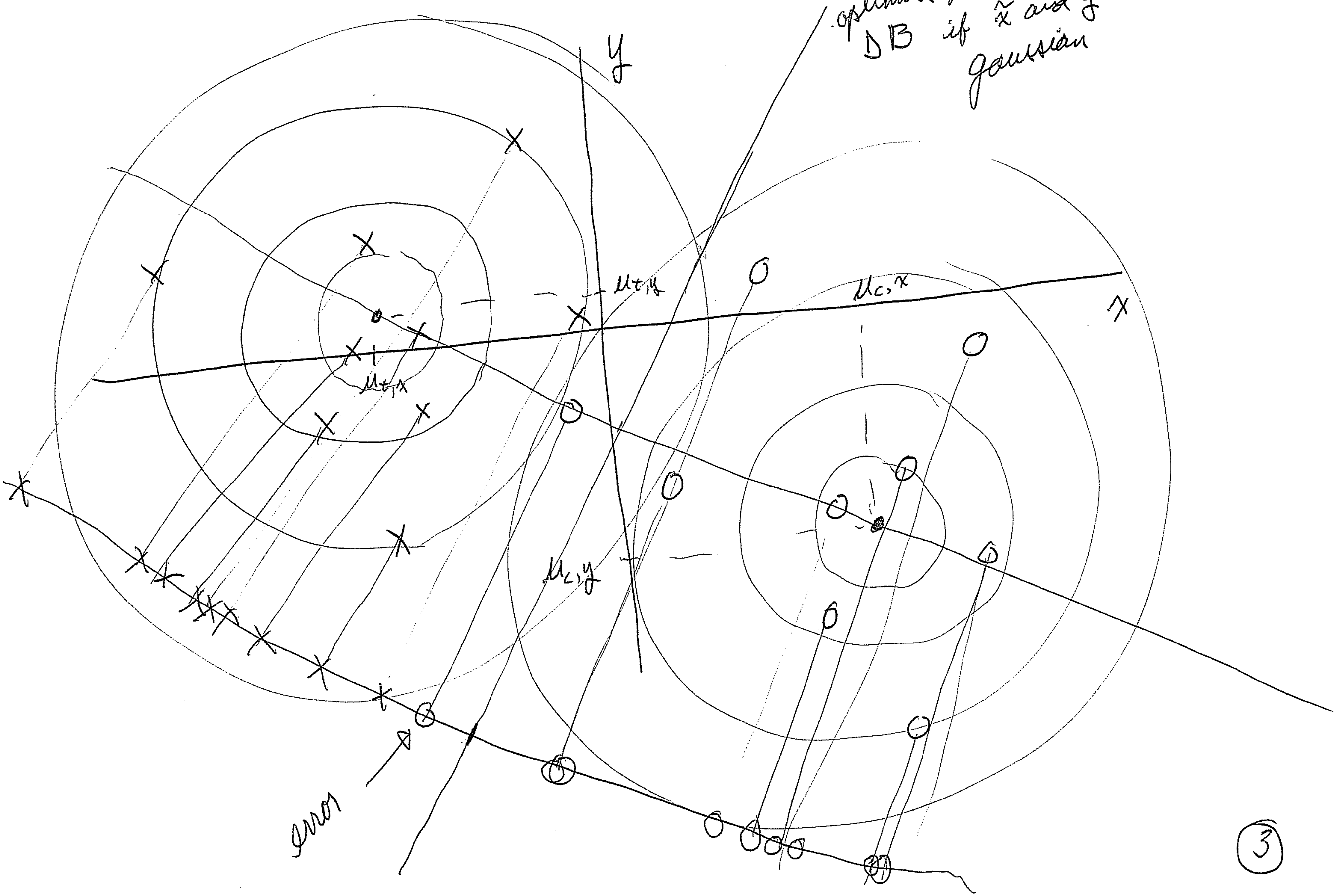
Optimum Decision Boundaries
Fisher Discriminant 3-24-05

Now consider a similar problem for 2-D data. The 2-D pdfs of each class can be represented by 3-D contours.



perspective view

Top View



optimum linear DB if $\tilde{x}$ and $\tilde{y}$ are iid gaussian
y
x
$\mu_{t,y}$
$\mu_{t,x}$
$\mu_{c,x}$
$\mu_{c,y}$
error

③

A linear DB is a straight line

By weighting the input dimensions properly, we can reduce N-D systems to be 1-D discriminants with linear DB optimum in MPE

Recall exponents for multi-variate gaussian random vectors:

For iid r.v. with $N(\underline{\mu}_i, \underline{I})$ as the distributions

Combine into the MLR

$$(\underline{x} - \underline{\mu}_t)^T \underline{I}^{-1} (\underline{x} - \underline{\mu}_t) - (\underline{x} - \underline{\mu}_c)^T \underline{I}^{-1} (\underline{x} - \underline{\mu}_c) \underset{H_t}{\overset{H_c}{\gtrless}} \eta$$

$$\cancel{\underline{x}^T \underline{x}} - 2\underline{x}^T \underline{\mu}_t + \underline{\mu}_t^T \underline{\mu}_t - \cancel{\underline{x}^T \underline{x}} + 2\underline{x}^T \underline{\mu}_c - \underline{\mu}_c^T \underline{\mu}_c \gtrless$$

$$\underline{x}^T \underbrace{2(\underline{\mu}_c - \underline{\mu}_t)}_{\text{let } \underline{a}} + \underbrace{\underline{\mu}_t^T \underline{\mu}_t - \underline{\mu}_c^T \underline{\mu}_c}_{b,} \gtrless \eta$$

For 2-D problem let $\underline{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$$x_1 a_1 + x_2 a_2 + b_1 \gtrless \eta$$

let $x_1 = x$, $x_2 = y$

Then we could rewrite as

$$x a_1 + y a_2 + b_1 \gtrless \eta.$$

and $y \gtrless x a + b$ where $b = \frac{\eta - b_1}{a_2}$

$a = \frac{a_1}{a_2}$

The problem is more complicated for different variances $\sigma_{x_1}^2 \neq \sigma_{x_2}^2$ and cross correlation $\rho \neq 0$

We can show that J is maximum if $\underline{w}$ satisfies $\underline{S}_B \underline{w} = \lambda \underline{S}_w \underline{w}$

if $|\underline{S}_w| \neq 0$ then $\underline{S}_w^{-1} \underline{S}_B \underline{w} = \lambda \underline{w}$

We know $\underline{S}_B \underline{w}$ is always in the direction of $\underline{m}_1 - \underline{m}_2$ so

$$\underline{S}_w^{-1} K (\underline{m}_1 - \underline{m}_2) = \lambda \underline{w}$$

$$\Rightarrow \underline{w} = S_w^{-1} (\underline{m}_1 - \underline{m}_2)$$

So the Fisher Discriminant is $y = \underline{w}^T \underline{x}$

There is, however an optimum solution for N-D spaces where both classes have identical covariance matrices. It is the Fisher Discriminant.

$$y = \underset{1 \times N}{\underline{w}^T} \; \underset{N \times 1}{\underline{x}}$$

transformation vector

assume $\underline{w}^T \underline{w} = 1$ (but magnitude is arbitrary)

we will show that $\underline{w} = S_w^{-1} (\underline{m}_1 - \underline{m}_2)$

where $\underline{m}_i = \dfrac{1}{N_i} \displaystyle\sum_{n=1}^{N_i} \underline{x}_{i,n}$

ith class $\quad i = 1$ or 2

where "n" is the nth sample vector

where $\underline{S}_i$ = scatter ~~matrix~~ matrix estimate of the ith class.

$$\underline{S}_i = \sum_{n=1}^{N_i} (\underline{x}_{i,n} - \underline{m}_i)(\underline{x}_{i,n} - \underline{m}_i)^T$$

The scatter matrix

$$\underline{S}_w = \underbrace{\underline{S}_1 + \underline{S}_2}_{\text{basically average of } \cancel{\text{inter}}\ \text{intra-class scatter matrix ?}} \propto$$

Covariance matrix estimate if the target and clutter have the same covariance matrix.

Let $\underline{S}_B = (\underline{m}_1 - \underline{m}_2)(\underline{m}_1 - \underline{m}_2)^T \equiv$ inter- ~~intra~~- class scatter ?

We define a criterion function

$$J(\underline{w}) = \left.\frac{\underline{w}^T \underline{S}_B \underline{w}}{\underline{w}^T \underline{S}_w \underline{w}}\right\} \text{Rayleigh quotient}$$

Consider a 1-D case

$$\underset{\sim}{S}_B = (m_1 - m_2)^2 = S_B$$

$$\underset{=}{S}_W = \sigma_1^2 + \sigma_2^2 = 2\sigma^2 \text{ for } \sigma_1^2 = \sigma_2^2$$

$\underline{w} = k \equiv$ arbitrary constant

$$w = \frac{(m_1 - m_2)^2}{2\sigma^2}$$