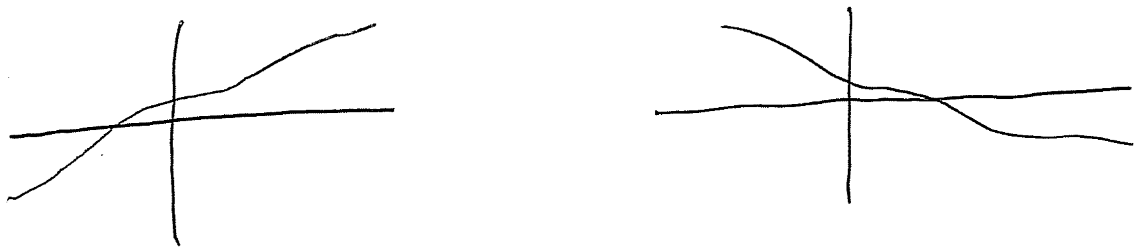
EE640 Lecture 9

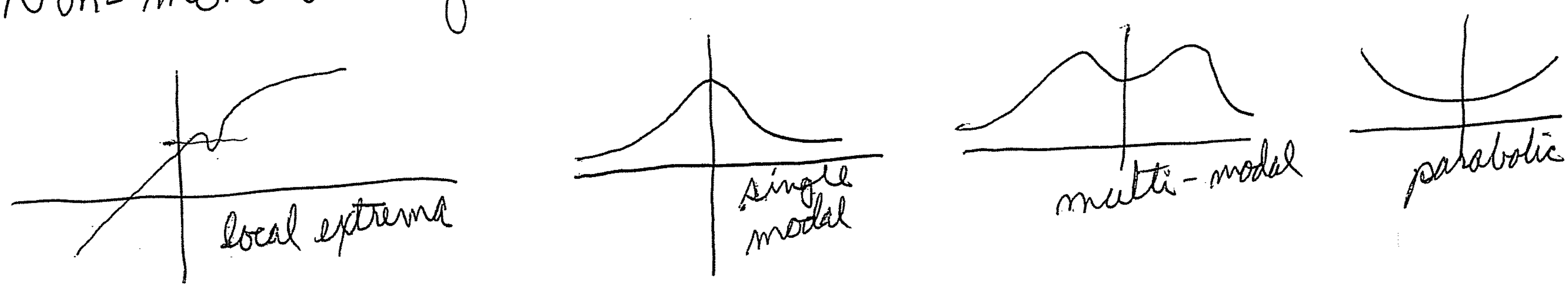
Functions of r.v.s

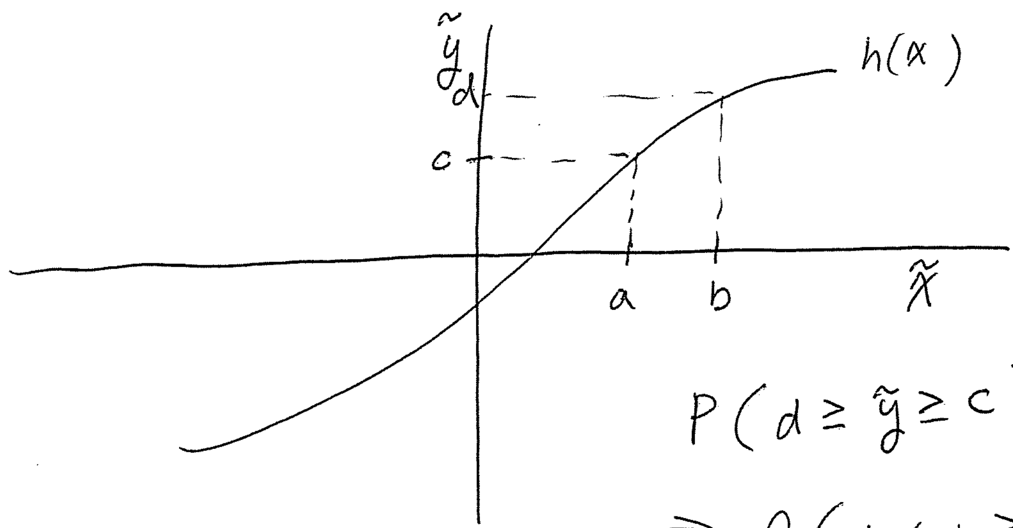
Given a function of one or more r.v.s, the functional value is also a r.v. We need to define or determine the resulting statistics given the function and input statistics

Note: Monotonic functions



Non-monotonic functions





$$P(d \ge \tilde{y} \ge c) = P(b \ge \tilde{x} \ge a)$$

$$\Rightarrow P(h(x) \ge \tilde{y}) = P(x \ge \tilde{x})$$

thus $\quad F_{\tilde{y}}(h(x)) = F_{\tilde{x}}(x)$

$$A = \int_{-\infty}^{h(x)} f_{\tilde{y}}(y)\,dy = \int_{-\infty}^{x} f_{\tilde{x}}(a)\,da = B$$

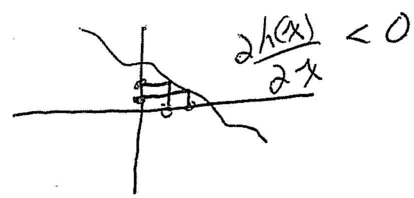
We want $f_{\tilde{y}}(y)$ in term of $f_{\tilde{x}}(x)$ and $h(x)$

Leibnitz Eq. $\quad \frac{\partial A}{\partial x} = \frac{\partial B}{\partial x}$

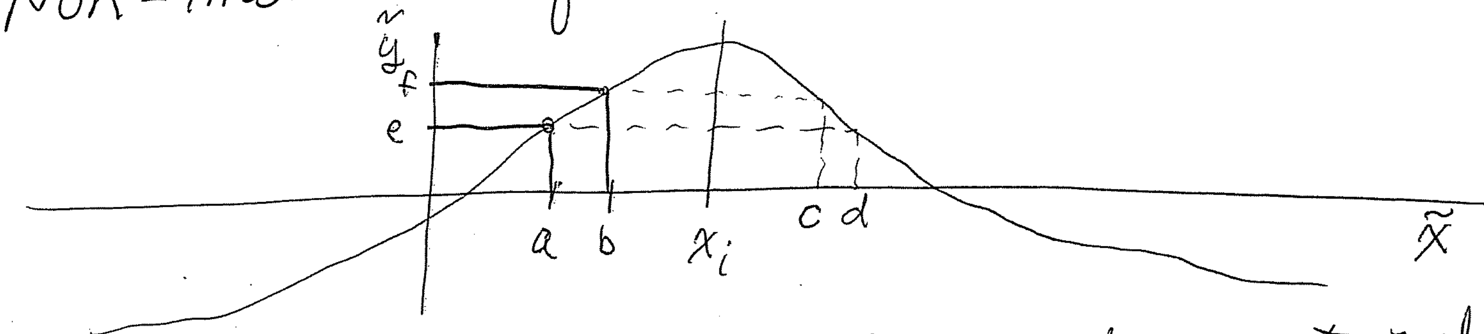
$$\frac{\partial h(x)}{\partial x} f_{\tilde{y}}(h(x)) = f_{\tilde{x}}(x) \cancelto{1}{\frac{dx}{dx}}$$

so $f_{\tilde{y}}(y) = \dfrac{f_{\tilde{x}}(x)}{\underbrace{\left|\dfrac{\partial h(x)}{\partial x}\right|}}$ where $x = h^{-1}(y)$

absolute sign comes from monotonically decreasing scenario



Non-monotonic functions

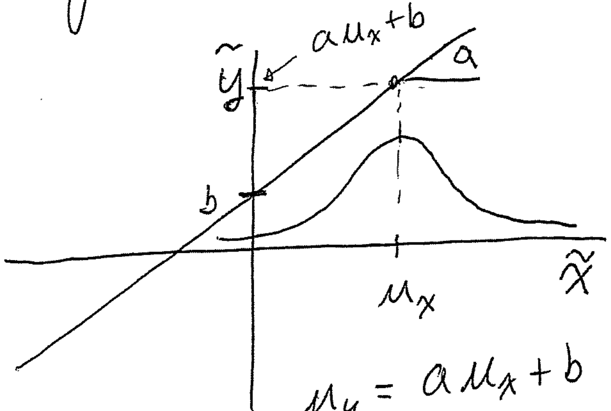


1. Piecewise model the function with monotonically increasing and decreasing functions seperated by extrema

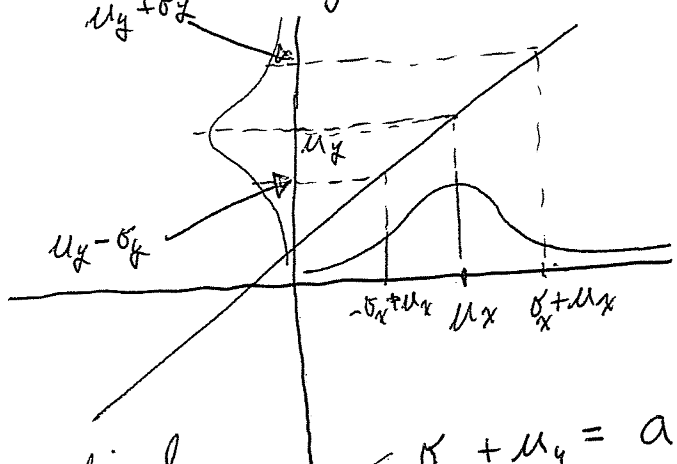
2. Sum multivalue probabilities

Ex: $P(f \geq \tilde{y} \geq e) = P(b \geq x \geq a) + P(d \geq x \geq c)$

Ex: given $\tilde{x} \sim N(\mu, \sigma^2)$ let $\hat{y} = a\tilde{x} + b$



$\mu_y = a\mu_x + b$



graphical method

some algebra

$$\begin{cases} \sigma_y + \mu_y = a(\sigma_x + \mu_x) + b \\ -\sigma_y + \mu_y = a(-\sigma_x + \mu_x) + b \end{cases}$$

$\sigma_y = a\sigma_x$

$h(x) = ax + b$

$x = \dfrac{y-b}{a} = h^{-1}(y)$

$\dfrac{\partial h(x)}{\partial x} = a$ we know $f_{\tilde{x}}(x) = \dfrac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}$

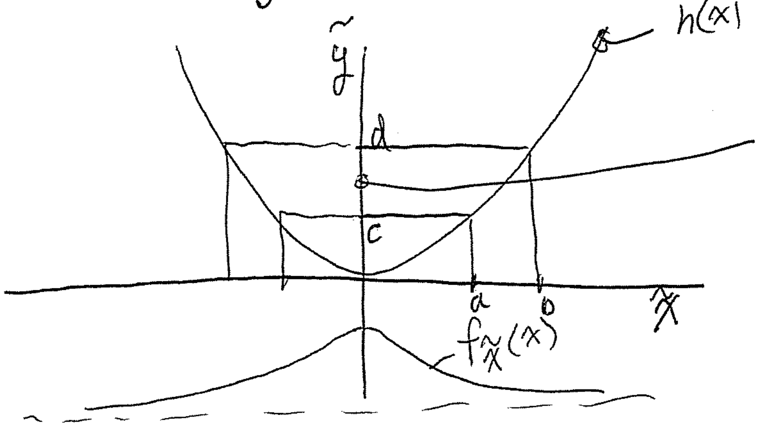
$$f_{\tilde{x}}(x)\Big|_{x=h^{-1}(y)} = \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{\left(\frac{y-b}{a} - \mu_x\right)^2}{2\sigma_x^2}}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{(y-(b+a\mu_x))^2}{2a^2\sigma_x^2}}$$

$$f_{\tilde{y}}(y) = \frac{f_{\tilde{x}}(h^{-1}(y))}{|a|} = \frac{1}{\sqrt{2\pi}\,(a\sigma_x)} e^{-\frac{(y-\mu_y)^2}{2(a\sigma_x)^2}}$$

where $\mu_y = a\mu_x + b$ and $\sigma_y^2 = a^2\sigma_x^2$

Ex: let $\tilde{y} = \tilde{x}^2$ let $\tilde{x} \sim N(0, \sigma^2)$



$P(c < y < d) = 2P(a < x < b)$

$$f_{\tilde{x}}(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}$$

for $x > 0$ $\quad f_{\tilde{y}}(y) = \left. \dfrac{f_x(h^{-1}(y))}{\left| \dfrac{\partial h(x)}{\partial x} \right|} \right|_{x = h^{-1}(y)}$

$h(x) = x^2$, $\quad x = \pm\sqrt{y}$

$\dfrac{\partial h(x)}{\partial x} = 2x$ and for $x > 0$ $\quad x = \sqrt{y}$

$$f_{\tilde{y}}(y) = 2\left(\frac{\frac{1}{\sqrt{2\pi}\,\sigma} e^{-y/2\sigma^2}}{2\sqrt{y}} \right) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\,\sigma\sqrt{y}} e^{-y/2\sigma^2} & \text{for } y \geq 0 \\ 0 & \text{for } y < 0 \end{cases}$$

double for negative x contribution

Functions of N variables

For the function of one r.v., in general we have

$$f_{\tilde{y}}(y) = \sum_{i=1}^{M} \left. \frac{f_{\tilde{x}}(x)}{\left| \frac{\partial h(x)}{\partial x} \right|} \right|_{x = x_i = h^{-1}(y)}$$

For N r.v.s we use the Jacobian determinant such that

$$J(\underline{x}) = \text{Det} \begin{bmatrix} \frac{\partial h_1(\underline{x})}{\partial x_1} & \frac{\partial h_1(\underline{x})}{\partial x_2} & \cdots & \frac{\partial h_1(\underline{x})}{\partial x_N} \\ \vdots & & & \vdots \\ \frac{\partial h_N(\underline{x})}{\partial x_1} & \cdots & \cdots & \frac{\partial h_N(\underline{x})}{\partial x_N} \end{bmatrix}$$

where $y_i = h_i(x_1, x_2 \ldots x_N)$ and $\underline{x} = [x_1\ x_2 \cdots x_N]^T$

so the multi-variant pdf is

$$f_{\underset{\sim}{y}}(\underline{y}) = \left. \frac{\sum_{i=1}^{m} f_{\underset{\sim}{\underline{x}}}(\underline{x})}{|J(\underline{x})|} \right|_{\underline{x} = \underline{x}_i = h_i^{-1}(\underline{y})}$$

Ex: $\tilde{z} = g(\tilde{x}, \tilde{y})$ $\quad \tilde{w} = h(\tilde{x}, \tilde{y})$

$$J(x, y) = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \underbrace{\begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}^{-1}}_{\text{trick } \# 1}$$

let $\tilde{r} = \sqrt{\tilde{x}^2 + \tilde{y}^2}$ $\qquad \underbrace{\tilde{\phi}}_{0 \le \phi \le 2\pi} = \arctan(\tilde{y}/\tilde{x})$

$\tilde{x}, \tilde{y} \sim N(0, \sigma^2)$

find $f_{\tilde{r}}(r)$, $f_{\tilde{\varphi}}(\varphi) = ?$

note: $\tilde{x} = \tilde{r} \cos \tilde{\varphi}$, $\tilde{y} = \tilde{r} \sin \tilde{\varphi}$

$$J(x,y) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix}^{-1} = \begin{vmatrix} \cos \varphi & -r \sin \varphi \\ \sin \varphi & r \cos \varphi \end{vmatrix}^{-1}$$

$$= (r \cos^2 \varphi + r \sin^2 \varphi)^{-1} = r^{-1} = 1/r$$

$$f_{\tilde{x}\tilde{y}}(x,y) = \frac{1}{2\pi\sigma^2} e^{\frac{(x^2+y^2)}{2\sigma^2}}$$

$$f_{\tilde{r}\tilde{\varphi}}(r,\varphi) = \left. \frac{e^{-\frac{(x^2+y^2)}{2\sigma^2}}}{2\pi\sigma^2 \left|\frac{1}{r}\right|} \right|_{r^2 = x^2+y^2} = \frac{r e^{-r^2/2\sigma^2}}{2\pi\sigma^2}$$

where
$0 < r < \infty$
$0 \le \varphi \le 2\pi$