EE640 Lecture 9A Auxillery Variable

Consider the case when you have less functions then r.v.s s.t.

$$\tilde{z} = h(\tilde{x}, \tilde{y}) = \tilde{x} + \tilde{y}$$

The Jacobian technique requires an equal number of functions and r.v.s. To satisfy this we define additional functions as auxillary variables s.t.

$$\tilde{w} = g(\tilde{x}, \tilde{y}) = \tilde{x}$$

The Jacobian is then

$$J(x,y) = \begin{vmatrix} \frac{\partial z}{\partial x} = 1 & \frac{\partial z}{\partial y} = 1 \\ \frac{\partial w}{\partial x} = 1 & \frac{\partial w}{\partial y} = 0 \end{vmatrix} = (0-1) = -1$$

This allows for determination of

$$f_{\tilde{z}\tilde{w}}(z,w) = \left.\frac{f_{\tilde{x}\tilde{y}}(x,y)}{|-1|}\right|_{\substack{x=w \\ y=z-x=z-w}}$$

$$= f_{\tilde{x}y}(w, z-w)$$

If $\tilde{x}$ and $\tilde{y}$ are independent then

$$f_{\tilde{z}\hat{w}}(z,w) = f_{\tilde{x}}(w)\, f_{\tilde{y}}(z-w)$$

so $$f_{\tilde{z}}(z) = \int_{-\infty}^{\infty} f_{\tilde{x}}(w)\, f_{\hat{y}}(z-w)\,dw$$

$$= f_{\tilde{x}}(z) * f_{\hat{y}}(z)$$

Let's look at this from the perspective of a cdf:

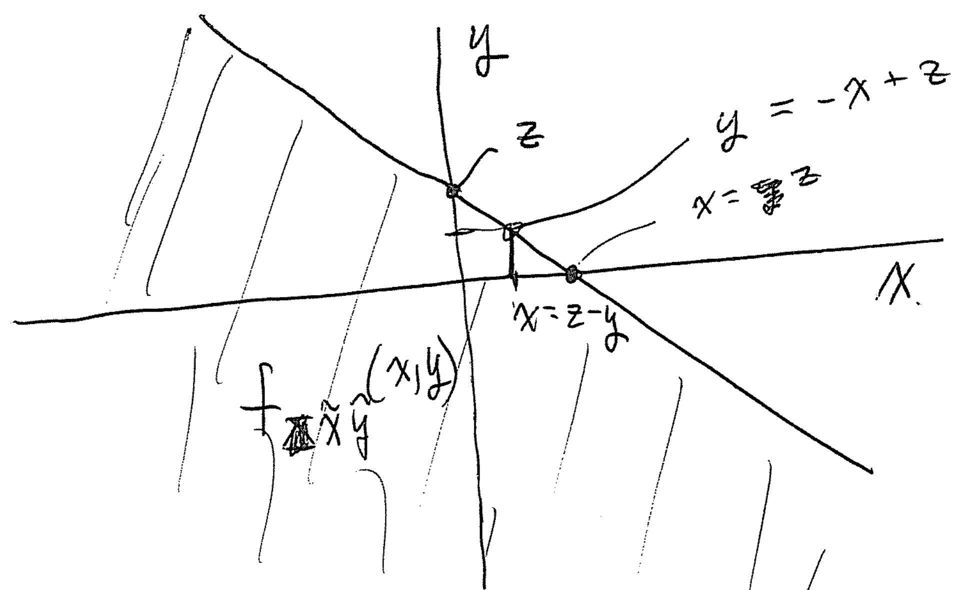
Sum of two independent r.v.s

$$\tilde{z} = \tilde{x} + \tilde{y}$$

$$F_{\tilde{z}}(z) = P(\tilde{z} < z)$$

The RI boundary is $z = x + y$

or $y = z - x = -x + z$



Use outer integration variable as y (4)

$$F_{\tilde{z}}(z) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{z-y} f_{\tilde{x}\tilde{y}}(x,y)\,dx \right] dy$$

$$F_{\tilde{z}}(z) = \int_{-\infty}^{\infty} \left[F_{\tilde{x}\tilde{y}}(z-y,\, y) - F_{\tilde{x}\tilde{y}}(-\infty, y) \right] dy$$

$$f_{\tilde{z}}(z) = \frac{\partial F_{\tilde{z}}(z)}{\partial z} = \int_{-\infty}^{\infty} \left[\frac{\partial F_{\tilde{x}\tilde{y}}(z-y,\, y)}{\partial z} - \underbrace{\cancel{\frac{\partial F_{\tilde{x}\tilde{y}}(-\infty, y)}{\partial z}}}_{0} \right] \cancel{dz}\, dy$$

$$= \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial z} \int_{-\infty}^{z-y} f_{\tilde{x}\tilde{y}}(x,y)\,dx \right] dy$$

$$= \int_{-\infty}^{\infty} f_{\tilde{x}\tilde{y}}(z-y,\, y)\,dy = \int_{-\infty}^{\infty} f_{\tilde{x}}(z-y) f_{\tilde{y}}(y)\,dy$$

$$f_{\tilde{z}}(z) = f_{\tilde{x}}(z) * f_{\tilde{y}}(z)$$

In general, for N independent variables s.t.

$$\tilde{z} = \sum_{n=1}^{N} \tilde{x}_n$$ where all $\tilde{x}_n$ are independent r.v.s

we can show

$$f_{\tilde{z}}(z) = f_{x_1}(z) * f_{x_2}(z) * \cdots * f_{\tilde{x}_N}(z)$$

EX: Square Law Selector

Let $\tilde{x}$ and $\tilde{y}$ be independent

Find $f_{\tilde{v}}(v)$ for $\tilde{z} = \tilde{x} + \tilde{y}$ and $\tilde{v} = \tilde{z}^2$

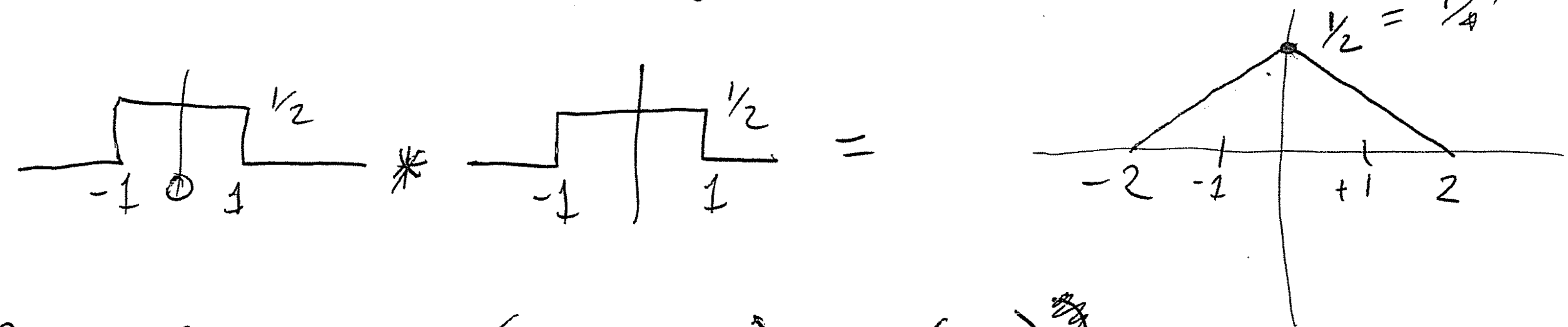
where $\tilde{x}, \tilde{y} \sim U(-1, 1)$ and iid

We know the pdfs are

$$f_{\tilde{x}}(x) = \tfrac{1}{2}\,\mathrm{rect}(x/2)$$
$$f_{\tilde{y}}(x) = \tfrac{1}{2}\,\mathrm{rect}(y/2)$$

(6)

We know $f_{\tilde{z}}(z) = \int_{-\infty}^{\infty} \mathrm{rect}\left(\frac{y}{2}\right) \mathrm{rect}\left(\frac{z-y}{2}\right) dy$



$\therefore f_z(z) = \frac{1}{4}\left(2 - |z|\right) \mathrm{rect}\left(\frac{z}{4}\right)$

$$f_{\tilde{v}}(v) = \left. \frac{\sum_{i=1}^{2} f_z(z_i)}{|2 z_i|} \right|_{z_i = \pm\sqrt{v}}$$

$\therefore f_{\tilde{v}}(v) = \frac{1}{4}\left(\frac{2}{2\sqrt{v}} - \frac{|\sqrt{v}|}{\sqrt{v}}\right) \mathrm{rect}\left(\frac{\sqrt{v}}{4}\right)$

$$f_{\tilde{v}}(v) = \begin{cases} \frac{1}{2}\left(\frac{2}{\sqrt{v}} - 1\right) & 0 < v < 4 \\ 0 & \text{else where} \end{cases}$$