EE640 Lecture 8 — Bivariate Gaussian r.v and correlation coefficient

①

Linearity of $E\{\ \}$

$$E\{a\tilde{x}+b\tilde{y}\} = \int_{-\infty}^{\infty}\int (ax+by)\, f_{\tilde{x}\tilde{y}}(x,y)\,dx\,dy$$

$$= a\int_{-\infty}^{\infty}\int x\, f_{\tilde{x}\tilde{y}}(x,y)\,dx\,dy + b\int_{-\infty}^{\infty}\int y\, f_{\tilde{x}\tilde{y}}(x,y)\,dx\,dy$$

$$= a\int_{-\infty}^{\infty} x \underbrace{\left[\int_{-\infty}^{\infty} f_{\tilde{x}\tilde{y}}(x,y)\,dy\right]}_{\text{marginal } f_{\tilde{x}}(x)} dx + b\int_{-\infty}^{\infty} y \underbrace{\left[\int_{-\infty}^{\infty} f_{\tilde{x}\tilde{y}}(x,y)\,dx\right]}_{f_{\tilde{y}}(y)} dy$$

$$= a\,E\{\tilde{x}\} + b\,E\{\tilde{y}\}$$

In general $E\left\{\sum_{i=1}^{N} \alpha_i \tilde{x}_i\right\} = \sum_{i=1}^{N} \alpha_i E\{\tilde{x}_i\}$

and

$$E\{E\{g(\tilde{x},\tilde{y})\mid\tilde{x}\}\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g(x,y)\,f(y\mid x)\,dy\right] f_{\tilde{x}}(x)\,dx$$

$$= \iint_{-\infty}^{\infty} g(x,y)\, f_{y\mid x}(y\mid x)\, f_x(x)\,dx\,dy$$

$$= \iint_{-\infty}^{\infty} g(x,y)\, f_{\tilde{x}\tilde{y}}(x,y)\,dx\,dy$$

$$= E\{g(\tilde{x},\tilde{y})\}$$

Correlation Coefficient

$$\rho_{xy} = \frac{\sigma_{xy}}{\sqrt{\sigma_x^2\,\sigma_y^2}} = \frac{\sigma_{xy}}{\sigma_x\,\sigma_y}$$

Let $\mu_x = E\{\tilde{x}\}$, $\mu_y = E\{\tilde{y}\}$

$$\sigma_x^2 = E\{(\tilde{x}-\mu_x)^2\} = \mathrm{Var}\{\tilde{x}\}$$

$$\sigma_y^2 = E\{(\tilde{y}-\mu_y)^2\}$$

$$\sigma_{xy} = E\{(\tilde{x}-\mu_x)(\tilde{y}-\mu_y)\}$$

<u>uncorrelated</u> $E\{\tilde{x}\tilde{y}\} = E\{\tilde{x}\}E\{\tilde{y}\}$

<u>orthogonal</u> $E\{\tilde{x}\tilde{y}\} = 0$

If $\tilde{x}$ is independent of $\tilde{y}$ then they are uncorrelated

If uncorrelated and $\mu_x = \mu_y = 0$ then orthogonal

so if $\rho_{xy} = 0$ then uncorrelated

if $\rho_{xy} = 1$ then $\sigma_{xy} = \sigma_x \sigma_y = \sigma^2$

Joint Gaussian pdf (bivariate Gaussian pdf)

$$f_{\tilde{x}\tilde{y}}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}$$

$$\times \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right]\right\}$$

Let $\rho = 0$ then

$$f_{\tilde{x}\tilde{y}}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y}\exp\left\{\frac{-1}{2}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right]\right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x}\exp\left\{\frac{-1}{2}\frac{(x-\mu_x)^2}{\sigma_x^2}\right\}\frac{1}{\sqrt{2\pi}\,\sigma_y}e^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}}$$

$$= f_{\tilde{x}}(x)\, f_{\tilde{y}}(y)$$

Conditional ~~Statistics~~ and Leibnitz Rule

Recall $F_{\tilde{x}}(x) = \sum_{i=1}^{n} F(x|A_i)\, P(A_i)$

Likewise

$$\frac{\partial F_{\tilde{x}}(x)}{\partial x} = f_{\tilde{x}}(x) = \sum_{i=1}^{n} \frac{\partial F(x|A_i)}{\partial x} P(A_i)$$

where

$$\frac{\partial F(x|A_i)}{\partial x} = \frac{\partial}{\partial x} \int_{-\infty}^{x} f(\lambda|A_i)\, d\lambda$$

We need Leibnitz rule which is

Given $G(u) = \int_{\alpha(u)}^{\beta(u)} H(x,u)\, dx$

then $$\frac{d\,G(u)}{du} = H(\beta(u),u)\,\frac{d\beta(u)}{du} - H(\alpha(u),u)\,\frac{d\alpha(u)}{du} + \int_{\alpha u}^{\beta(u)} \frac{\partial H(x,u)}{\partial u}\,dx$$

so applying this to $\dfrac{d\,F(x|Ai)}{dx}$

$$= f(x|Ai).1 - f(-\infty|A_i)\underbrace{\frac{d\infty}{dx}}_{0} + \int_{-\infty}^{x} \underbrace{\frac{d\,f(x|Ai)}{dx}}_{0}\,dx$$

$$= f(x|Ai)$$

$$\therefore \quad f_{\tilde{x}}(x) = \sum_{i=1}^{n} f_{\tilde{x}}(x|A_i)\,P(A_i)$$