EE630 Lecture 14 (Chapter 5)

Transform Analysis of LTI Systems

Frequency Response

\[ Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega}) \]

\[ |Y(e^{j\omega})| = |H(e^{j\omega})| \cdot |X(e^{j\omega})| \]

\[ \angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega}) \]

Ideal LP and HP filters

\[ H_{LP} = \sum_{k} \text{rect} \left( \frac{\omega + k2\pi}{2\omega_c} \right) \]

\[ h_{LP}[n] = \frac{\sin \omega_c n}{\pi n} \]
\[ H_{hp}(e^{j\omega}) = \sum_{k} \text{rect} \left( \frac{\omega + k2\pi + \pi}{2(\pi - \omega_c)} \right) \]

\[ H_{hp}'(e^{j\omega}) = 1 - H_{LP}(e^{j\omega}) \]

\[ h_{hp}[n] = s[n] - \sin \frac{\omega_c n}{\pi} \]

**Phase Distortion and Delay**

\[ h_{id}[n] = s[n-n_d] \quad H_{id}(e^{j\omega}) = e^{-j\omega n_d} \]

\[ |H_{id}(e^{j\omega})| = 1 \quad \angle H_{id}(e^{j\omega}) = -\omega n_d \quad |\omega| < \pi \]
Ideal Linear Phase LPF

\[ H_{LP}(e^{j\omega}) = e^{-j\omega nd} \operatorname{rect} \left( \frac{\omega}{2\omega_c} \right) \text{ for } |\omega| < \pi \]

or \[ h_{LP}[n] = \sin \omega_c (n - nd) \frac{1}{\pi (n - nd)} \]

\( nd \) is the group delay in sample units.

In general the group delay is

\[ \tau(\omega) = \text{grd} \left[ H(e^{j\omega}) \right] = -\frac{d}{d\omega} \angle H(e^{j\omega}) \]

**System Functions from Difference Equations**

\[ \sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{m} b_k x[n-k] \]

\[ \sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{m} b_k z^{-k} X(z) \]
\[ H(z) = \sum_{k=0}^{m} b_k z^{-k} \]
\[ \frac{\sum_{k=0}^{N} a_k z^{-k}}{\prod_{k=0}^{N} (1-d_k z^{-1})} = \left( \frac{b_0}{a_0} \right) \prod_{k=1}^{m} (1-c_k z^{-1}) \]

Zeros at \( z = c_k \) and poles at \( z = 0 \)
Poles at \( z = d_k \) and a zero at \( z = 0 \)

Ex: \( H(z) = \frac{(1+z^{-1})^2}{(1-\frac{1}{2}z^{-1})(1+\frac{3}{4}z^{-1})} \)

\[ H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}} = \frac{Y(z)}{X(z)} \]

\[ y[n] + \frac{1}{4} y[n-1] - \frac{3}{8} y[n-2] = x[n] + 2x[n-1] + x[n-2] \]
Stability and Causality

If the system is causal then \( h(n) \) is

right-handed seq.

\[ \text{ROC is outer most pole} \]

Assume stable then \( h(n) \) is absolutely summable.

\[ \sum_{n} |h(n)| < \infty \iff \sum_{n} |h(n)| e^{-n} < \infty \]

For \( |z| = 1 \) then for stability has ROC include

the system is stable if the ROC

includes the unit circle.
Ex: Find ROC
\[ y[n] = \frac{1}{2} y[n-1] + y[n-2] = x[n] \]

t hen \[ H(z) = \frac{1}{1 - \frac{1}{2} z^{-1} + z^{-2}} = \frac{1}{(1 - \frac{1}{2} z^{-1})(1 - z^{-1})} \]

if causal \( |z| > 2 \), not stable

if stable \( \frac{1}{2} < |z| < 2 \)

not stable or causal
\[ \text{ROC } \Rightarrow |z| < \frac{1}{2} \]

Inverse System
For LTI system
\[ G(z) = \frac{1}{H(z)} \quad H_i(z) = 1 \]

where \( H_i(z) = \frac{1}{H(z)} \) so in the DT domain
\[ g[n] = h[n] \ast h_i[n] = s[n] \]
and so \( H_i(e^{j\omega}) = \frac{1}{H(e^{j\omega})} \) for \( |H(e^{j\omega})| > 0 \)

Note: Deconvolution \( Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega}) \)

If we know \( H(e^{j\omega}) H_i(e^{j\omega}) = 1 \)

then \( Y(e^{j\omega}) H_i(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega}) H_i(e^{j\omega}) = X(e^{j\omega}) \)

Rationale system functions

\[
H(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{m} (1-c_k z^{-1})}{\prod_{k=1}^{N} (1-d_k z^{-1})} \quad H_i(z) = \frac{a_0}{b_0} \frac{\prod_{k=1}^{N} (1-d_k z^{-1})}{\prod_{k=1}^{N} (1-c_k z^{-1})}
\]

poles \( \rightarrow \) zeros but ROCs must overlap
Ex: \( H(z) = \frac{1 - 0.5 z^{-1}}{1 - 0.9 z^{-1}} \) with ROC \(|z| > 0.9\)

\[ H_i(z) = \frac{1 - 0.9 z^{-1}}{1 - 0.5 z^{-1}} \] with ROC choice of \(|z| > 0.5\) to overlap

so \[ h_i[n] = 0.5^n u[n] - 0.9 (0.5)^{n-1} u[n-1] \]

Impulse Response for Rational systems

Given rational function of \( z^{-1} \) and first-order poles

\[ H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_r}{1 - d_r z^{-1}} \]

\underline{long division} \quad M \geq N

\underline{PFE}

If causal then ROC is outside all poles
\[ h[n] = \sum_{r=0}^{M-N} B_r s[n-r] + \sum_{k=1}^{N} A_k d_k^n u[n] \]

There are 2 classes

(IIR) Class 1: At least one nonzero pole is not cancelled by a zero.
The \( A_k (d_k)^n u[n] \) will be finite length.
These are infinite impulse response (IIR) filters.

Ex: \[ y[n] - a \cdot y[n-1] = x[n] \]
\[ H(z) = \frac{1}{1 - a z^{-1}} \]

ROC \( |z| > |a| \)

stable if \( |a| < 1 \)
\[ h[n] = a^n u[n] \]
(FIR)
(Class 2: $H(z)$ has no poles except at $z=0$)

At $N=0$:

$$ H(z) = \sum_{k=0}^{M} b_k z^{-k} $$

Assuming $b_0 = 1$:

$$ h[n] = \sum_{k=0}^{M} b_k \delta[n-k] = \begin{cases} b_0 & 0 \leq n \leq M \\ 0 & \text{else} \end{cases} $$

This is called finite impulse response (FIR)

Ex: $h[0] = 1$
\[ H(z) = 1 - \alpha \frac{1}{1 - \alpha^2} \frac{z^{m+1}}{z + a} \]

\[ H(z) = \frac{1}{m \alpha^2 + \alpha + 1} \]

\[ \text{let } z = k e^{\frac{\pi}{2m+1} \phi} \]

\[ k = 1, 2, \ldots \]

\[ \begin{array}{c}
  k = 0 \\
  k = 1 \\
  \vdots \\
  k = 0 \\
  \end{array} \]
\[ k = 2 \]

\[ \frac{1}{e^{i\phi(m+z) - 1}} \]

\[ e^{i\phi - 1} = \theta \]

\[ \theta = (m+\epsilon) \]
\[ y[n] - ay[n-1] = x[n] - a^{m+1} x[n-m-1] \]

zeros \( z_k = a e^{j2\pi k/(m+1)} \) for \( k = 0, 1, \ldots, m \)

for \( m = 7 \)

\[ y[n] = \sum_{k=0}^{M} a^k x[n-k] \]

no zero cancelled by a pole

7th order pole @ 0
Frequency Response of a Rational System

If a stable LTI is rational then

\[ H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}} \]

in terms of poles and zeros

\[ H(e^{j\omega}) = \frac{\delta_0}{\delta_0} \frac{\prod_{k=1}^{M} (1-c_k e^{-j\omega})}{\prod_{k=1}^{N} (1-d_k e^{-j\omega})} \]

Approximating frequency response by looking at the z-domain w/respect to unit circle.
\[ H(\omega) = 1 - \text{re} e^{i\omega \theta} = \frac{\omega - \text{re} e^{i\omega}}{\omega} \quad r < 1 \]

\[ \hat{\omega}_3 = \hat{\omega}_1 - \hat{\omega}_2 \]
\[ \hat{\omega}_1 = e^{i\omega} \]
\[ \hat{\omega}_2 = \text{re} e^{i\omega} \]
\[ \hat{\omega}_3 = e^{i\omega} - \text{re} e^{i\omega} \]

\[ |H(e^{i\omega})| = \left| 1 - \text{re} e^{i\omega} e^{-i\omega} \right| = \left| \frac{e^{i\omega} - \text{re} e^{i\omega}}{e^{i\omega}} \right| \]

\[ = \left| \frac{\hat{\omega}_3}{\hat{\omega}_1} \right| \]

\[ |\hat{\omega}_1| = 1 \quad \therefore \quad |H(e^{i\omega})| = \left| e^{i\omega} + e^{i\omega} \theta \right| \]

\[ \angle \left( 1 - \text{re} e^{i\omega} e^{-i\omega} \right) = \angle \hat{\omega}_3 - \angle \hat{\omega}_1 = \theta_3 - \omega \]
where \( \hat{N}_3 \) is a zero vector and \( \hat{N}_i \) is a pole vector.