QMF Analysis and Synthesis Design

Consider the two-channel QMF bank:

\[ x[n] \xrightarrow{\text{analysis}} x_{a0} \xrightarrow{H_0(z)} x_{o0} \xrightarrow{\downarrow 2} \hat{x} \]

\[ x_{a1} \xrightarrow{H_1(z)} \xrightarrow{\downarrow 2} x_{o1} \xrightarrow{G_1(z)} \downarrow 2 \xrightarrow{\text{synthesis}} \hat{x}[n] \]

We would like to design the analysis and synthesis so that:
\[ \hat{x}[n] = x[n-T] \] where \( T \) is latency.

Consider first the DFT formulation:

\[ X_{a_0}(e^{j\omega}) = \frac{1}{2} \left[ X(e^{j\omega/2}) H_0(e^{j\omega/2}) + X(e^{j(\omega-2\pi)/2}) H_0(e^{j(\omega-2\pi)/2}) \right] \]

see Eq. 11.2.13

\[ X_{a_1}(e^{j\omega}) = \frac{1}{2} \left[ X(e^{j4\omega/2}) H_1(e^{j\omega/2}) + X(e^{j(\omega-2\pi)/2}) H_1(e^{j(\omega-2\pi)/2}) \right] \]

(aligning)
The output of the synthetic section has

\[ \hat{X}(e^{j\omega}) = \hat{X}_{a0}(e^{j\omega})G_0(e^{j\omega}) + \hat{X}_{a1}(e^{j\omega})G_1(e^{j\omega}) \]

Now substitute \( \hat{X}_{a0}(\cdot) = \hat{X}_{s0}(\cdot) \), \( \hat{X}_{a1}(\cdot) = \hat{X}_{s1}(\cdot) \)

\[ \hat{X}(e^{j\omega}) = \frac{1}{2} \left[ H_0(e^{j\omega})G_0(e^{j\omega}) + H_1(e^{j\omega})G_1(e^{j\omega}) \right] \hat{X}(e^{j\omega}) \]

\[ + \frac{1}{2} \left[ H_0(e^{j(-\omega-n)})G_0(e^{j\omega}) + H_1(e^{j(-\omega-n)})G_1(e^{j\omega}) \right] \hat{X}(e^{j(\omega-n)}) \]

The z-transform

\[ \hat{X}(z) = \frac{1}{2} \left[ H_0(z)G_0(z) + H_1(z)G_1(z) \right] \hat{X}(z) \]

\[ + \frac{1}{2} \left[ H_0(-z)G_0(z) + H_1(-z)G_1(z) \right] \hat{X}(-z) \]

\[ = G(z) \hat{X}(z) + A(z) \hat{X}(-z) \]
We would like \( A(z) = 0 \) s.t.
\[
H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0
\]

In the DT frequency domain
\[
H_0(e^{j\omega} - \pi)G_0(e^{j\omega}) + H_1(e^{j(\omega-\pi)})G_1(e^{j\omega}) = 0
\]

This condition can be satisfied by
\[
G_0(e^{j\omega}) = H_1(e^{j(\omega-\pi)}) \quad \text{and} \quad G_1(e^{j\omega}) = -H_0(e^{j\omega})
\]

low part

\[
\text{high part}
\]

where \( H_0(e^{j\omega}) \) is LP and \( H_1(e^{j\omega}) \) is HP.

Since they are "mirror" filters
\[
H_0(e^{j\omega}) = H(e^{j\omega}) \quad \text{and} \quad H_1(e^{j\omega}) = H(e^{j(\omega-\pi)})
\]

so \( G_0(e^{j\omega}) = H_0(e^{j\omega}) \) and \( G_1(e^{j\omega}) = -H_1(e^{j\omega}) \)
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In z-plane

\[ H_0(z) = H(z), \quad H_1(z) = H(-z) \]

\[ G_0(z) = H_0(z) \quad \text{and} \quad G_1(z) = -H_1(z) \]

Condition for perfect Reconstruction

\[ \hat{X}(n) = x(n-k) \]

We have \( A(z) = 0 \) but we require

\[ Q(z) = \frac{1}{2} \left[ H_0(z) G_0(z) + H_1(z) G_1(z) \right] = z^{-k} \]

such that \( H(z) - H^2(-z) = 2z^{-k} \)

or in DFT

\[ H^2(e^{j\omega}) - H^2(e^{j(\omega - \pi)}) = 2e^{-j\omega k} \]

Then

\[ |H^2(e^{j\omega}) - H^2(e^{j(\omega - \pi)})| = C = 2 \quad \text{then} \quad \hat{X}(n) = X(n-k) \]

If \( H(e^{j\omega}) \) satisfies the magnitude condition and has linear phase

Then \( \hat{X}(n) = X(n-k) \). But there could be a solution without linear phase.