We know that there is considerable analog filter design, so we would like to make use of that in digital filter design.

i.e., \( h[n] = h_a(nT) \) for \( n = 0, 1 \ldots \)

Recall for \( x_3(t) = x_a(t) \sum_{n=0}^{N} \delta(t-nT) \)

\[ X_3(f) = \frac{1}{T} \sum_{k} X_a(f-kT) \]

A aliasing occurs if

\[ X_a(f) \quad X_a(f-\frac{1}{T}) \]
\[ H(z) = \frac{1}{1 - \sum_{k=1}^{\infty} z^{-k}} \]

Impulse invariance is inappropriate for high-pass design due to aliasing problem.

Design process: map s-plane to z-plane mapping.

\[ \frac{1}{1 - sT} \]

Design process:

let

\[ z = re^{j\omega}, \quad e^{\omega T} = e^{\omega T} \]

for \( r < 1 \), \( \omega < \pi \)

\[ I = 1, \quad 0, \quad 1 \]

\[ h[n] e^{-nT} \]

\[ h = \Delta T \]

\[ \omega = \Delta T \]

\[ 0 < 0, \quad 0 < 1 \]

\[ \sum \]
Consider wrapping

\[ H_a(s) = \sum_{k=1}^{N} \frac{C_k}{s - f_k} \]

\[ h_a(t) = \sum_{k=1}^{N} C_k e^{f_k t} \]

It can be shown \( H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{f_k T} z^{-1}} \)
Ex: Convert \( H_a(s) = \frac{s + 0.1}{(s + 0.1)^2 + 9} = \frac{s + 0.1}{(s + 1) + j3)(s + 1) - j3) \)

zero \( s = -0.1 \) and \( p_k = -0.1 + j3 \)

\[ r = e^{sT} = e^{-0.1T} \]

\[ H(z) = \sum_{k=0}^{\infty} \frac{c_k}{1 - e^{-0.1T}z^{-1}} = \frac{1}{1 - e^{-0.1T}e^{j3T}z^{-1}} + \frac{1}{1 - e^{-0.1T}e^{-j3T}z^{-1}} \]
\[ H(z) = 1 - \frac{\left( e^{-0.1T} \cos 3T \right) z^{-1}}{1 - \left( 2 e^{-0.1T} \cos 3T \right) z^{-1} + e^{-0.2T} z^{-2}} \]
Consider the Analog and digital design specifications such that (see text)

\[ H(\text{Re}^s) \]

\[ H(\text{Im}^s) \]

\[ 0 \leq \omega \leq \omega_p \]

\[ \omega_p = 0.2\pi \]

\[ \omega_s = 0.3\pi \]

\[ 1 - s_1 = 0.89125 \]

\[ s_2 = 0.17783 \]

\[ T_d = 1 \quad \text{then} \quad \omega = 0 \quad \Rightarrow \quad 0.89125 \leq |H_c(\omega)| \leq 1 \quad |\omega| < 0.2\pi \]

Assume Butterworth filter

\[ |H_c(\text{Re}^s)|^2 = \frac{1}{1 + (\frac{\omega}{\omega_c})^{2N}} \]

\[ 0.3\pi < |\omega| \leq \pi \]
Butterworth is monotonic to

\[ 1 + \left(\frac{0.2\pi}{\omega_c}\right)^{2N} = \left(\frac{1}{0.89125}\right)^2 \quad \text{-- A} \]

\[ 1 + \left(\frac{0.3\pi}{\omega_c}\right)^{2N} = \left(\frac{1}{0.17783}\right)^2 \quad \text{-- B} \]

Take the log

\[ \log \left(\frac{0.2\pi}{\omega_c}\right)^{2N} = \log \left(A - 1\right) \]

Likewise

\[ 2N \left(\log 0.2\pi - \log \omega_c\right) = \log \left(A - 1\right) \]

\[ 2N \left(\log 0.3\pi - \log \omega_c\right) = \log \left(B - 1\right) \]

\[ 2N \left(\log 0.2\pi - \log 0.3\pi\right) = \log \left(A - 1\right) - \log \left(\beta - 1\right) \]

Solve for \(N\) then round \(N\) up to nearest integer.

Solve for \(\omega_c\) knowing \(N\).
Find pole locations of Butterworth poles

$$H_c(s) H_c(-s) = \frac{1}{1 + (s/\omega_0)^2N}$$

Poles are uniformly distributed on a circle of radius $\omega_c$.

$s$-plane

let $N = n$

The discrete time filter is

$$H(z) = \sum_{k=1}^{N} \frac{T_A A_R}{1 - e^{sA_D}z^{-1}}$$

Conclusion conjugate term.