Hilbert Transformer

A Hilbert transform is based on the function

\[ H_\alpha(f) = \begin{cases} \Im & 0 < f \\ 0 & f < 0 \end{cases} \]

For example, consider the CT quadratic detector with Hilbert transform.

If \( x(t) = A \cos(2\pi f_c t + \theta) \) then the envelope

\[ x_d(t) = A \cos(2\pi f_c t) \cos(2\pi f_c t + \theta) \]
and quadrature term is

\[ X_q(t) = A \sin (2\pi f_c t) \cos (2\pi f_c t + \theta) \]

Using \( \cos A \cos B = \frac{\cos(A - B) + \cos(A + B)}{2} \)

and \( \sin A \sin B = \frac{\sin(A + B) - \sin(A - B)}{2} \)

to yield

\[ X_I(t) = \frac{A}{2} \left( \cos \theta + \cos(4\pi f_c t + \theta) \right) \]

and

\[ X_q(t) = \frac{A}{2} \left( -\sin \theta + \sin(4\pi f_c t + \theta) \right) \]

The LP filters are designed to remove the \( 2f_c \) terms such that

\[ g_I(t) = \frac{A}{2} \cos \theta \quad \text{and} \quad g_q(t) = -\frac{A}{2} \sin \theta \]
We can find phase and amplitude from $g_\pm(t)$ and $g_q(t)$.

$$\theta = \arctan \left( \frac{-g_\pm(t)}{g_q(t)} \right) = \arctan \left( \frac{\sin \delta}{\cos \delta} \right)$$

Amplitude?

$$\frac{A^2}{4} = \left( g_\pm(t) \right)^2 + \left( g_q(t) \right)^2 = \frac{A^2}{4} \cos^2 \delta + \frac{A^2}{4} \sin^2 \delta = \frac{A^2}{4}$$

So

$$A = 4 \sqrt{\left( g_\pm(t) \right)^2 + \left( g_q(t) \right)^2}$$

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The Hilbert transform was simply used to correct a cosine to sine. But it is usually used to analyze single sideband modulation. The DT Hilbert T.

$$Hd(e^{j \omega}) = \begin{cases} -j & 0 < \omega < \pi \\ j & -\pi < \omega < 0 \end{cases}$$

And

$$h_d[n] = \begin{cases} \frac{1}{2\pi} \int_{-\pi}^{\pi} Hd(e^{j \omega}) e^{j \omega n} \, d\omega & n \neq 0 \\ \frac{\sin \left( \frac{n \pi}{2} \right)}{n} & n = 0 \end{cases}$$