EE636: Lecture 16

DFT properties

Circular Shift

Both DT and Fourier domains are periodic

so $x[n] = x[n \pm mN]$ for $m = 0, 1, \ldots$

and N is the sequence length or number of points

and $X[k] = X[k \pm mN]$

Symmetry

If $x[n]$ is real then $\text{Real}\{X[k]\} = \text{Real}\{X[N-k]\}$

$N-k$? given a 4 pt. seq then $X[k]$ indices are

$0 \quad 1 \quad 2 \quad 3 \leftarrow N-1$

N is even

dc

unique values

equal real values

if N is odd then the symmetry is

$$\begin{array}{ccccc} & & N-2 & N-1 & \\ 0 & 1 & 2 & 3 & 4 \\ \uparrow & & & & \\ dc & & & & \end{array} \qquad N=5$$

and $\operatorname{Imag}\{X[k]\} = -\operatorname{Imag}\{X[N-k]\}$

or in summary $X[k] = X^*[N-k]$ for $x[n]$ real.

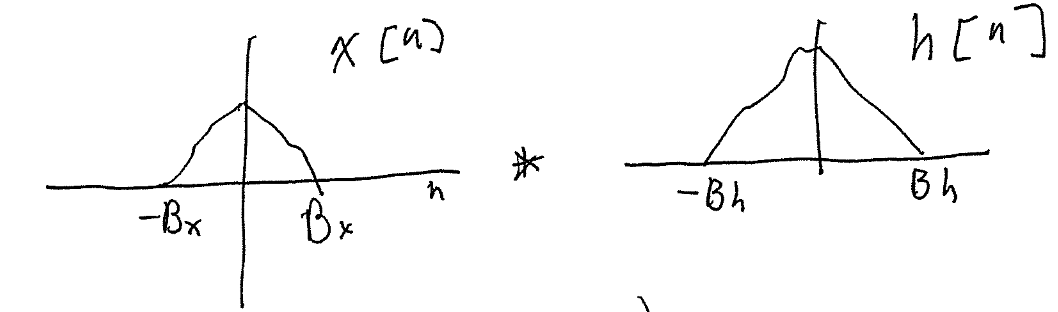
If $x[n]$ is real and even, i.e. $x[n] = x^*[n]$ and $x[n] = x[N-n]$

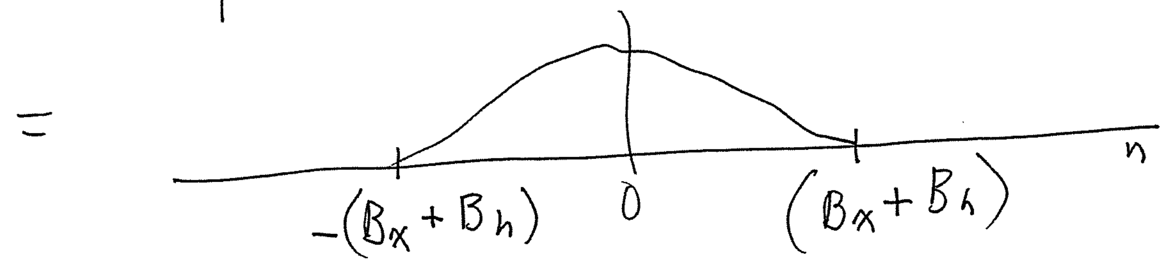
then $X[k]$ is also real and even.

Circular versus linear convolution

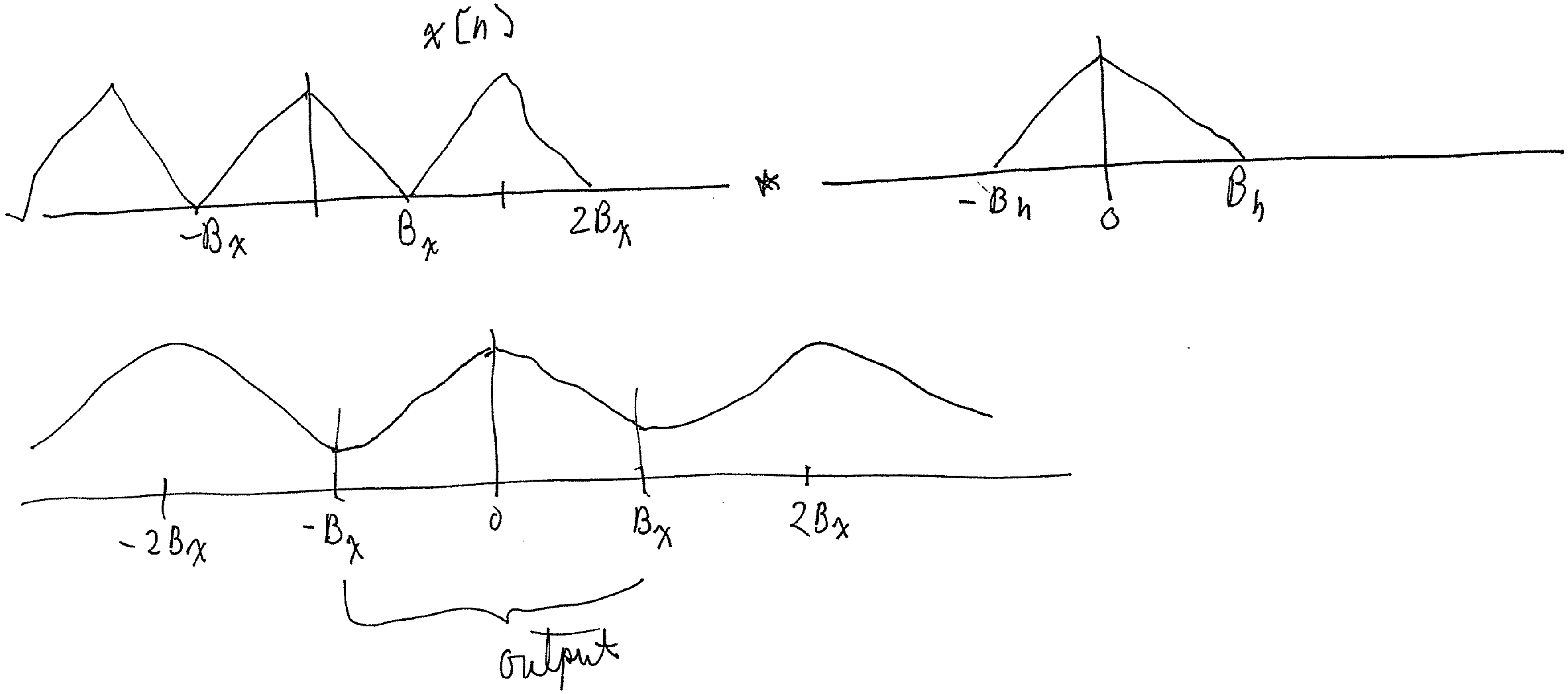
linear convolution

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m]\, h[n-m]$$

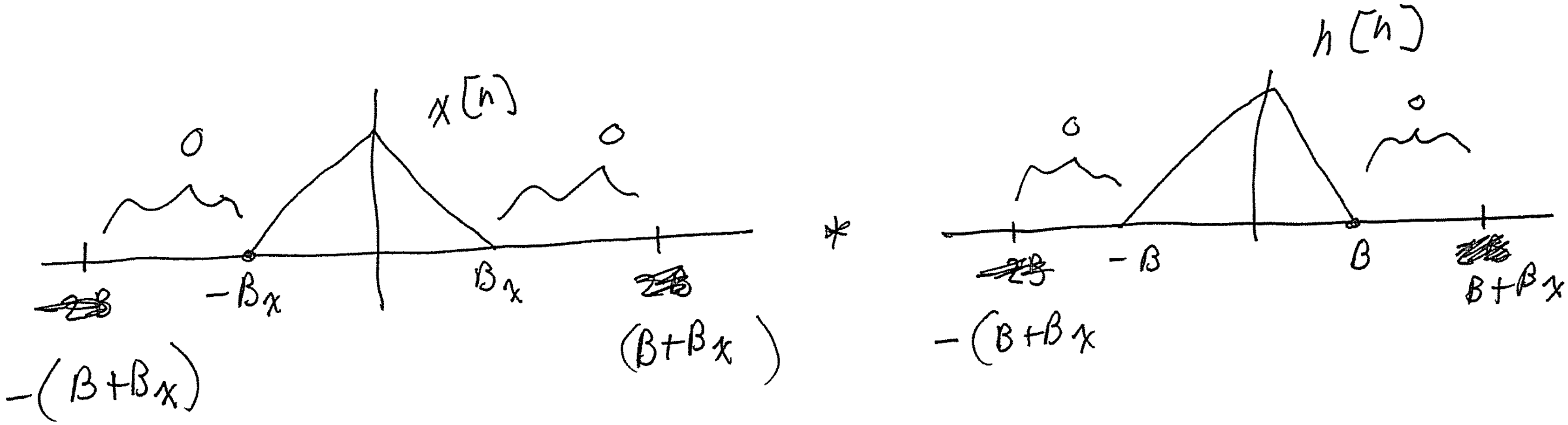


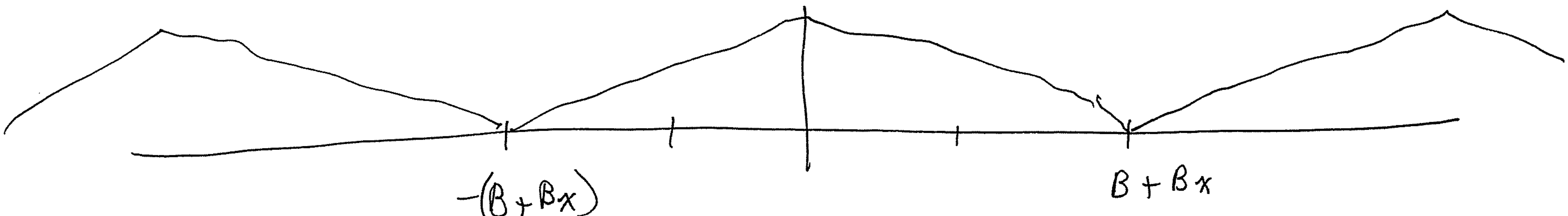


Circular Convolution $y[n] = x[n] \circledast h[n] = \sum_{m=0}^{N-1} x[m]\,h[n-m]$



Simulating linear convolution with zero padding





identical to linear convolution.

Circular convolution in frequency domain

$$Y[k] = X[k]H[k] \xleftrightarrow{DFT^{-1}} y[n] = x[n] ⊛ h[n]$$

Leakage

"Leakage" is an error introduced by the finite length of a DFT

Consider $x[n] = \cos 2\pi \frac{k_c n}{N}$

where the window width is N,

and k_c may be a none integer value

Consider the ~~full range of such that~~ CT model of this as

$$s_s(t) = \text{rect}\left(\frac{t}{T}\right) \cos(2\pi f_c t)\, P_{T/N}(t)$$

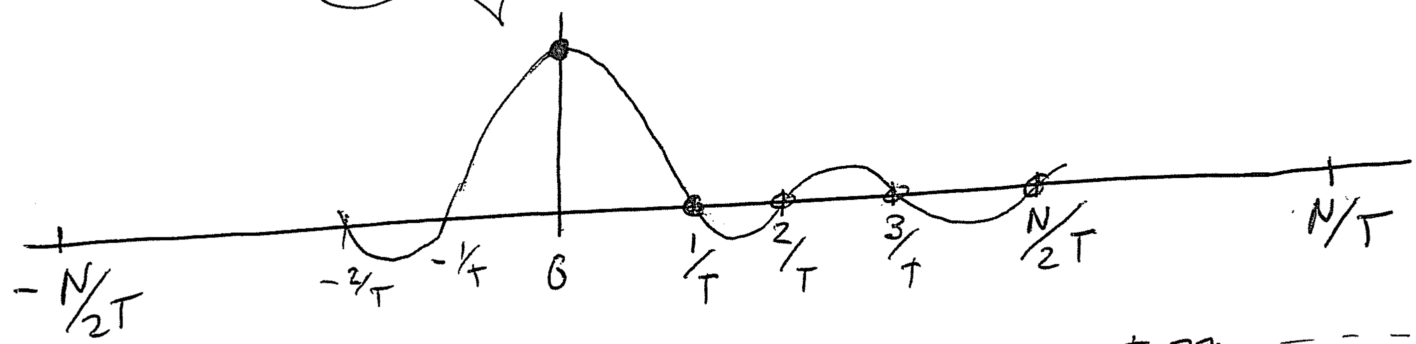
where $P_{T/N}(t) = \sum_n \delta(t - nT/N)$

The FT is $S_s(f) = T\,Sa(\pi f T) * \mathcal{F}\{\cos\} * \frac{N}{T} \sum_k \delta\left(f - \frac{kN}{T}\right)$

so $S_s(f) = N \, Sa(\pi f T) * \mathcal{F}\{\cos\} *$

let $k = 0$ for just one period

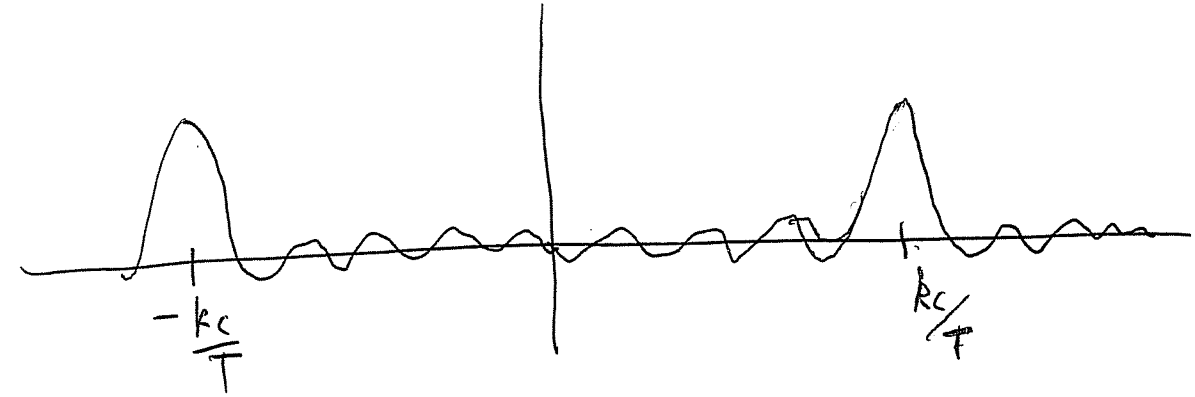
$$S(f) = \frac{N}{2} \underbrace{Sa(\pi f T)} * \frac{\delta(f - f_c) + \delta(f + f_c)}{2}$$



nulls are at $\pi f T = \pm\pi, \pm 2\pi, \pm 3\pi$ - - - -

$$f_{null} = \pm \frac{m}{T}$$

if $f_c = \frac{k_c}{T}$ ~~then~~



and $S[k] \propto \dfrac{\delta[k-k_c] + \delta[k+k_c]}{2}$

The nulls do not contribute between coefficients

BUT! if $f_c \neq \frac{m}{T}$ then k_c does not line up and we get contributions between samples