DFT properties

Circular Shift

Both DT and Fourier domains are periodic

so \( x[n] = x[n + mN] \) for \( m = 0, 1, \ldots \)

and \( N \) is the sequence length or number of points

and \( X[k] = X[k + mN] \)

Symmetry

If \( x[n] \) is real then \( \text{Real} \{X[k]\} = \text{Real} \{X[N-k]\} \)

\( N-K \)? given a 4pt. seq then \( X[k] \) indices are

\( N \) is even

\( k \) is unique value

\( N \) is odd

\( k \) equal real values
if \( N \) is odd then the symmetry is
\[
0 \quad 1 \quad 2 \quad 3 \quad 4 \quad N-1 \quad N-2
\]
and \( \text{Im} \{ X[k] \} = -\text{Im} \{ X[N-k] \} \)

or in summary \( X[k] = X^*[N-k] \) for \( X[n] \) real.

If \( x[n] \) is real and even, i.e. \( x[n] = x^*[n] \) and \( x[n] = x^*[N-n] \)
then \( X[k] \) is also real and even.

Circular versus linear convolution

Linear convolution
\[
y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m] h[n-m]
\]
Circular Convolution \( y[n] = x[n] \ast h[n] = \sum_{m=0}^{N-1} x[m] h[n-m] \)
Simulating linear convolution with zero padding.

Circular convolution in frequency domain:

\[ Y[k] = X[k] * H[k] \xrightarrow{\text{DFT}^{-1}} y[n] = x[n] \circledast h[n] \]
Leakage

"Leakage" is an error introduced by the finite length of a DFT

Consider \( X[n] = \cos \left( \frac{2\pi kn}{N} \right) \)

where the window width is \( N \)

and \( k \) can be an integer value

Consider the full-range of \( \theta \) such that CT model of this as

\[ s_s(t) = \text{rect} \left( \frac{t}{T} \right) \cos \left( 2\pi f_c t \right) p_{TN}(t) \]

where \( p_{TN}(t) = \sum_n s(t - nTN) \)

The FT is

\[ S_s(f) = T S_a(f + f_c) \sum_k \sum_n \cos \left( \frac{2\pi kn}{N} \right) \sum_{n} s(t - \frac{kn}{T}) \]
So \( S_s(f) = N S_a(\pi f T) * \frac{\pi}{T} \cos^2 \frac{\pi}{2} \left( \frac{\pi}{2} f T \right) \)

Let \( k = 0 \) for just one period

\[
S_s(f) = \frac{N}{2} S_a(\pi f T) * \left( S(f - f_c) + S(f + f_c) \right)
\]

The nulls are at \( \pi f T = \pm \pi, \pm 2\pi, \pm 3\pi \)

\[
f_{null} = \frac{m}{T}
\]

If \( f_c = \frac{f_s}{T} \) then
\[
S[k] \propto \sqrt{S[k-k_c] + S[k+k_c]}
\]

The nulls do not contribute between coefficients.

BUT, if \( f_c < \frac{M}{T} \) then \( k_c \) does not line up and we get contributions between samples.