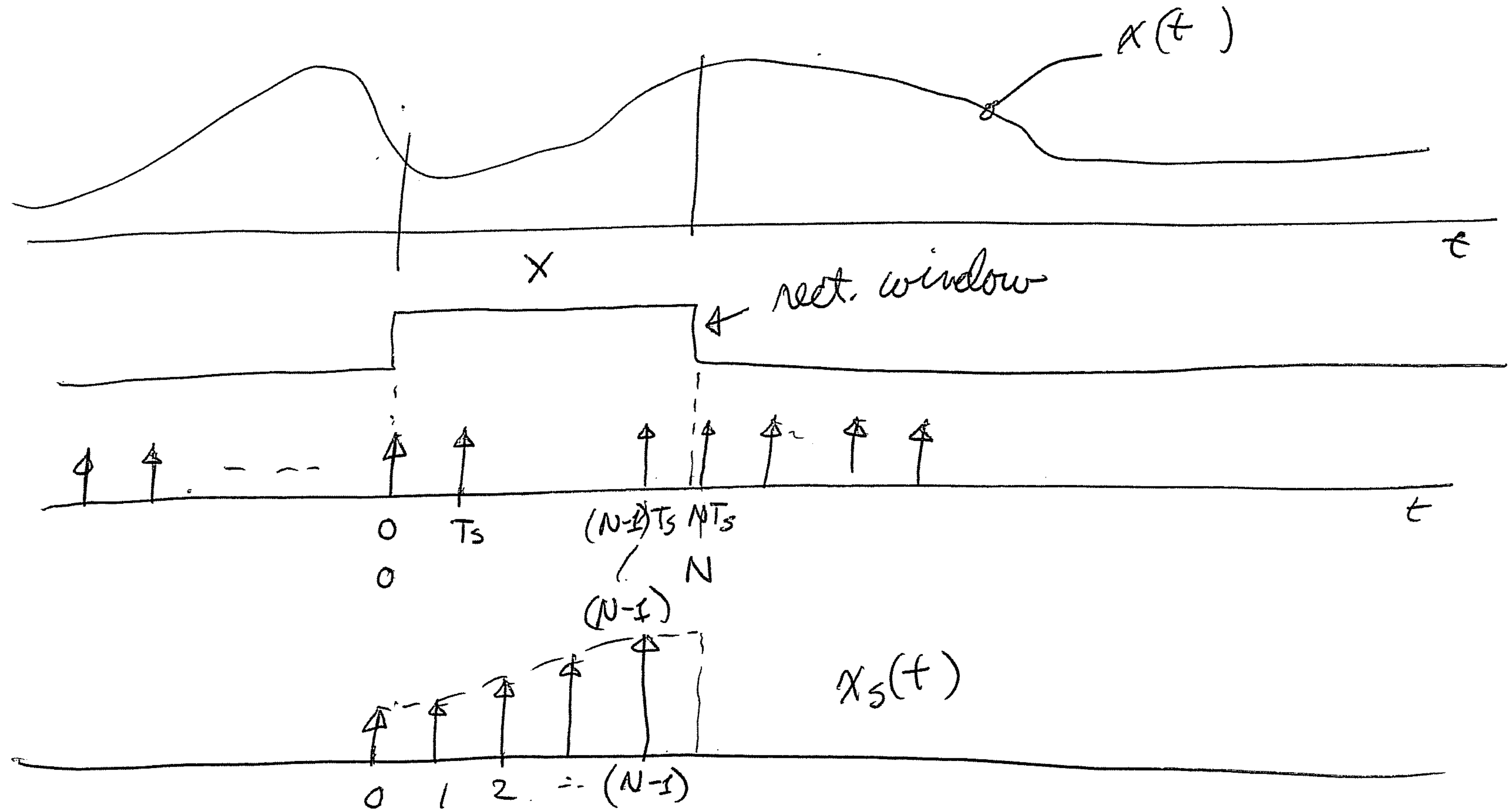
EE630: Lecture 15

Discrete Fourier Transform (DFT)

The DFT requires a finite length sequence $x[n]$ of length N. The analogy is to take a DTFT of a periodic signal $x[n] = x[n+N]$ and sample the frequencies $\omega_k = \frac{2\pi k}{N}$ for $k = 0, 1, 2, \ldots (N-1)$ resulting in N Fourier coefficients denoted by $X[k]$ for $k = 0, 1, \ldots (N-1)$

We have sampling in both domains and both domains are periodic by N.

To show that



That is

$$x_s(t) = \mathrm{rect}\left(\frac{t - NT_s/2}{NT_s}\right) x(t) \sum_n \delta(t - nT_s)$$

where NT_s is the window width.

$$x_s(t) = \text{rect}(\quad) \sum_n \underbrace{x[n]}_{x(nT_s)} \delta(t - nT_s)$$

$$= \sum_{n=0}^{N-1} x[n]\, \delta(t - nT_s)$$

CTFT of $x_s(t)$

$$\mathcal{F}\{x_s(t)\} = \int_{-\infty}^{\infty} \sum_{n=0}^{N-1} x[n]\, \delta(t - nT_s)\, e^{-j2\pi f t}\, dt$$

$$X_s(f) = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi f nT_s}$$

Now let's sample the frequency values at

$f = k/NT_s$ for $k = 0, 1, \cdots N-1$

$$X[k] = X\left(\frac{k}{NT_s}\right) = \sum_{n=0}^{N-1} x[n]\, e^{-j 2\pi \frac{k}{N\cancel{T_s}} n \cancel{T_s}}$$

$$\boxed{X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi kn}{N}}} \Rightarrow \text{DFT of } x[n]$$

The inverse DFT is

$$\boxed{x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi kn}{N}}} \quad \text{inverse DFT}$$

Matrix form of DFT

We rewrite the DFT as

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn} \quad \text{for } k = 0, 1 \ldots (N-1)$$

and $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, W_N^{-kn}$ where $W_N = e^{-j 2\pi / N}$

so $W_N^{kn} = e^{-j 2\pi kn/N}$

let the vector $\underline{x} = [x[0] \; x[1] \; \cdots \; x[N-1]]^T$

$\underline{\hat{x}} = [X[0] \; X[1] \; \cdots \; X[N-1]]^T$

Let $\underbrace{\varphi_k}_{\text{Fourier Vector}} = [W_N^0 \; W_N^k \; W_N^{2k} \; \cdots \; W_N^{(N-1)k}]^T$

Let $\underbrace{\underline{\underline{\Phi}}}_{\text{DFT matrix}} = [\varphi_0 \; \varphi_1 \; \cdots \; \varphi_{N-1}]$.

It can be shown $\underline{\underline{\Phi}}^T = \underline{\underline{\Phi}}$ and $\underline{\underline{\Phi}}^{-1} = \frac{1}{N} \underline{\underline{\Phi}}^*$

by using $\varphi_k^T \varphi_m^* = \sum_{n=0}^{N-1} e^{-j 2\pi \frac{(k-m)n}{N}} = N \delta[k-m]$

We can write $\underbrace{\underline{\hat{x}}^T}_{\text{DFT}} = \underline{x}^T \underline{\underline{\Phi}} \Rightarrow \underbrace{\boxed{\underline{\hat{x}} = \underline{\underline{\Phi}}\, \underline{x}}}_{\text{DFT}}$

The DFT^{-1} is $\boxed{\underline{x} = \frac{1}{N} \underline{\underline{\Phi}}^{*} \underline{\hat{x}}}$