The DFT requires a finite length sequence \( x[n] \) of length \( N \). The analogy is to take a DFT of a periodic signal \( x[n] = x[n+N] \) and sample the discrete Fourier coefficients \( X[k] \) for \( k = 0, 1, 2, \ldots, (N-1) \) resulting in a periodic signal \( X[k+N] = X[k] \). We have periodicity in both domain and both domains.
To show that

That is

\[ X_5(t) = \text{rect} \left( \frac{t - NT_s/2}{NT_s} \right) x(t) \leq x(t - nT_s) \]

where \( NT_s \) is the window width.
\[ X_s(t) = \text{rect}(t) \sum_{h=0}^{N-1} x[n] \delta(t - nT_s) \]

CTFT of \( X_s(t) \)

\[ \mathcal{X}_s(f) = \int_{-\infty}^{\infty} X_s(t) e^{-j2\pi ft} dt = \sum_{h=0}^{N-1} x[n] e^{-j2\pi fnT_s} \]

Now let's sample the frequency values at

\[ f = \frac{k}{NT_s} \text{ for } k = 0, 1, \ldots, N-1 \]
\[ X[k] = X \left( \frac{\theta_k}{NT_0} \right) = \sum_{h=0}^{N-1} x[h] e^{-j2\pi \frac{k}{N} h} \]

\[ x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi \frac{kn}{N}} \quad \text{inverse DFT} \]

Matrix form of DFT

We rewrite the DFT as

\[ X[k] = \sum_{h=0}^{N-1} x[h] W_N^{kh} \quad \text{for } k = 0, 1, \ldots, (N-1) \]

and

\[ x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \quad \text{where } W_N = e^{-j\frac{2\pi}{N}} \]
\[ W_N^k = e^{-\frac{j}{2} \pi \frac{k}{N}} \]

Let the vector \( \hat{x} = [x_0 \ x_1 \ \ldots \ x_{N-1}]^T \)

\( \hat{x} = [x_0 \ x_1 \ \ldots \ x_{N-1}]^T \)

Let \( \Phi_k = [W_N^0 \ W_N^1 \ W_N^{2k} \ \ldots \ W_N^{(N-1)k}]^T \)

\( \Phi = \begin{bmatrix} \Phi_0 & \Phi_1 & \cdots & \Phi_{N-1} \end{bmatrix} \)

\( \Phi^T \Phi \) matrix

It can be shown \( \Phi^T = \Phi \) and \( \Phi^{-1} = \frac{1}{N} \Phi^* \)

By using \( \Phi_k \Phi_m = \sum_{n=0}^{N-1} e^{-\frac{j}{2} \pi \frac{(k-m)n}{N}} = N \delta_{k-m} \)

We can write \( \hat{x}^T = \frac{1}{\Phi} x \Rightarrow \hat{x} = \Phi x \)
The DFT is

\[ X = \frac{1}{N} \Phi^* \hat{X} \]