Lecture 6 Inverse Z-form

Contour Integration

Consider two types of Z-form

Bilateral $\quad Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

unilateral $\quad Z_1\{x[n]\} = \sum_{n=0}^{\infty} x[n] z^{-n}$

We will always discuss the bilateral Z-form

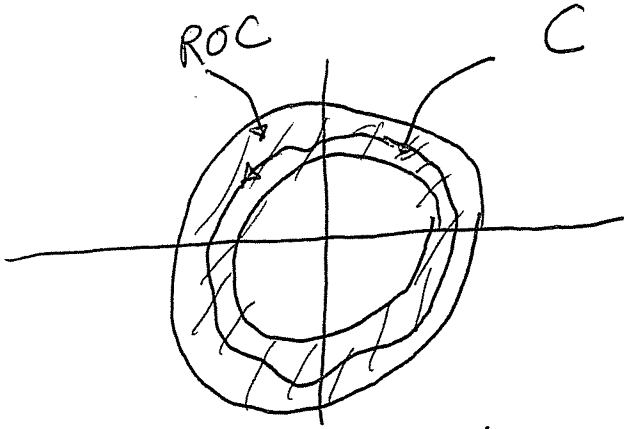
The inverse Z-form

given $X(z) = \sum_{n} x[n] z^{-n}$

we have $\quad x[n] = \frac{1}{2\pi j} \underbrace{\oint_C X(z) z^{n-1} dz}_{\text{contour integration}}$

Description of contour integral

$$\oint_C X(z)\, z^{n-1}\, dz = \oint_C \sum_k \underbrace{x(k)\; z^{n-1-k}}_{\text{converges on } C} dz$$



$$= \sum_k x(k) \oint_C z^{n-1-k}\, dz$$

The Cauchy integral theorem

$$\frac{1}{2\pi j} \oint_C z^{n-1-k}\, dz = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases} = \delta[n-k]$$

where C must enclose the origin

Thus, $2\pi j \sum_k x[k]\,\delta[n-k] = x[n]\,2\pi j$

so $$x[n] = \frac{1}{2\pi j} \oint_C X(z)\, z^{n-1}\, dz$$

Properties

Combined Z-forms have an ROC that is the intersection of all subcomponent ROCs.

Inverse z-form by contour integration

Cauchy residue theorem

$$\frac{1}{2\pi j}\oint_C \frac{f(z)}{(z-z_0)^k}\,dz = \begin{cases} \dfrac{1}{(k-1)!}\,\dfrac{d^{k-1} f(z)}{dz^{k-1}}\Bigg|_{z=z_0} & z_0 \text{ inside } C \\ 0 & \text{if } z_0 \text{ outside } C \end{cases}$$

$$x[n] = \sum_{\text{All poles } \{z_i\} \text{ inside } C} \left[\text{residues of } X(z)\, z^{n-1} \text{ @ } z = z_i\right]$$

$$= \sum_{\text{all } i} \underbrace{(z-z_i)\, X(z)\, z^{n-1}\Big|_{z=z_i}}_{\text{provided the poles are simple}}$$

EX: $\underline{X}(z) = \dfrac{1}{1-az^{-1}} \quad |z|>|a|$

$z_0 = a z^{-1}$

$$x[n] = \frac{1}{2\pi j} \oint_C \frac{z^{n-1}}{1-az^{-1}}\,dz = \frac{1}{2\pi j} \oint_C \underline{X}(z) z^{n-1}\,dz$$

$$= \frac{1}{2\pi j} \oint_C \frac{z^n}{z-a}\,dz = \frac{1}{2\pi j} \oint_C P(z)\,dz \quad |C|>|a|$$

let $P(z) = \dfrac{f(z)}{g(z)}$ where $f(z)$ has no poles inside C

$g(z)$ has simple roots

1. For $n \geq 0$ $f(z) = z^n$ only has zeros

$$\frac{1}{2\pi j} \oint_C \frac{z^n}{z-a}\,dz = \sum_{k=1} \frac{1}{1!} f(z)\Big|_{z=a} = f(a) = a^n$$

for $n \geq 0$ $x[n] = a^n$

2. for $n < 0$ $f(z) = z^n$ has n order poles at $z=0$ which is inside C

let $n=-1$

$$x[-1] = \frac{1}{2\pi j} \oint_C \frac{1}{z(z-a)}\, dz$$

The residue at $z=0$ $f(z) = \dfrac{1}{z-a}$

$k=1$ $f(z)\Big|_{z=0} = -1/a$

residue @ $z=a$ $f(z) = \dfrac{1}{z}$ $k=1$ $x[n] = -1/a + 1/a = 0$

$f(z)\Big|_{z=a} = 1/a$

$n=-2$ $$x[-2] = \frac{1}{2\pi j} \oint_C \frac{1}{z^2(z-a)}\, dz$$

$z_0 = 0$ $f(z) = \dfrac{1}{z-a}$ $k=2$ $\Rightarrow \dfrac{1}{1!} \dfrac{d}{dz}\left(\dfrac{1}{z-a}\right)\Big|_{z=0} = -1/a^2$

$z_0 = a \qquad f(z) = \frac{1}{z^2} \quad k = 1$

$\left.\frac{1}{z_0^2}\right|_{z_0 = a} = 1/a^2 \qquad x[-2] = -1/a^2 + 1/a^2 = 0$

and so on for $n = -3, -4 \ldots\ldots -\infty$

Thus, $x[n] = a^n u[n]$