Consider two types of $z$-form:

Bilateral: $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

Unilateral: $X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$

We will always discuss the bilateral $z$-form.

The inverse $z$-form:

Given $X(z) = \sum_{n} x(n) z^{-n}$

we have $x(n) = \frac{1}{2\pi j} \oint_X(z) z^{n-1} dz$

contour integration
Description of contour integral
\[ \oint \frac{X(z)}{z^{n-1}} \, dz = \sum_{k} X(k) \oint \frac{1}{z^{n-1-k}} \, dz \]
Converges on \( C \)
\[ = \sum_{k} X(k) \oint \frac{1}{z^{n-1-k}} \, dz \]

The Cauchy integral theorem
\[ \frac{1}{2\pi i} \oint \frac{1}{z^{n-1-k}} \, dz = \sum_{k=0}^{n} \left\{ \begin{array}{ll} -1 & \text{if } k = n \\ 0 & \text{if } k \neq n \end{array} \right. \]

where \( C \) must enclose the origin
Thus, \[ x[k] \leq \sum_{k} x[n-k] = x[n] \leq \frac{1}{2\pi j} \oint_{C} X(z) z^{-n-1} \, dz \]

**Properties**

Combined \( z \)-forms have an ROC that is the intersection of all subcomponents ROCs.
Inverse Z-transform by contour integration

Cauchy residue theorem

\[
\frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^k} \, dz = \begin{cases} 
\frac{1}{(k-1)!} \left. \frac{d^{k-1} f(z)}{dz^{k-1}} \right|_{z=z_0} & \text{if } z_0 \text{ inside } C \\
0 & \text{if } z_0 \text{ outside } C
\end{cases}
\]

\( x[n] = \sum \left[ \text{residue of } X(z) z^{-n-1} \right] \text{ at } z = z_i \)

All poles \( \{z_i\} \) inside \( C \)

\[
= \sum \sum (z - z_i) \bar{X}(z) z^{-n-1} \left| \begin{array}{c}
\text{provided the poles are simple} \\
Z = z_i
\end{array} \right|
\]
EX: \( X(z) = \frac{1}{1-a z^{-1}} \quad |z| > |a| \)

\( z_0 = a z^{-1} \)

\[ X[n] = \frac{1}{2\pi i} \oint_C \frac{z^{n-1}}{1-a z^{-1}} \, dz = \frac{1}{2\pi i} \oint_C \frac{X(z) z^{n-1}}{z} \, dz \]

\[ = \frac{1}{2\pi i} \oint_C \frac{z^n}{z-a} \, dz = \frac{1}{2\pi i} \oint_C \frac{P(z) \, dz}{|C| > |a|} \]

let \( P(z) = \frac{f(z)}{g(z)} \) where \( f(z) \) has no poles inside \( C \)

\( g(z) \) has simple roots

1. For \( n > 0 \) \( f(z) = z^n \) only has zeros

\[ \frac{1}{2\pi i} \oint_C \frac{z^n}{z-a} \, dz = \sum_{k=1}^{\frac{1}{f(z)}} \frac{1}{z=a} \]

for \( n > 0 \) \( x[n] = a^n \)
2. For $n < 0$, $f(z) = z^n$ has $n$ other poles at $z = 0$ which is inside $C$

Let $n = -1$

$$\kappa[-1] = \frac{1}{2\pi i} \oint_C \frac{1}{z(z-a)} \, dz$$

The residue at $z = 0$ $f(z) = \frac{1}{z-a}$

$$k = \frac{f(z)}{z = 0} = -\frac{1}{a}$$

Residue @ $z = a$ $f(z) = \frac{1}{z}$ $k = 1$ $\kappa[n] = -\frac{1}{a} + \frac{1}{a} = 0$

$$f(z) \Big|_{z = a} = \frac{1}{a}$$

$n = -2$

$$\kappa[-2] = \frac{1}{2\pi i} \oint_C \frac{1}{z^2(z-a)} \, dz$$

$z_0 = 0$ $f(z) = \frac{1}{z-a}$ $k = 2$ $\Rightarrow \frac{1}{1!} \left. \frac{d}{dz} \left( \frac{1}{z-a} \right) \right|_{z = 0} = \frac{-1}{a^2}$
$$z_0 = a \quad \frac{f(z)}{2} = \frac{1}{2} \quad k=1$$

$$\frac{1}{2} = \frac{1}{2} \quad x = \mathbb{L} \quad x \mathbb{L} = -\frac{1}{2} \quad y = \frac{1}{2}$$

$$0 = b \quad y = \mathbb{L}$$

And so on for $n = -3, -4, \ldots, -\infty$

Thus, $x[n] = a^n u[n]$