Lecture 3: EE630

Discrete - Time Linear System

We will need to know some geometric series identities

$$\sum_{n=0}^{\infty} r^n = 1 + \sum_{n=1}^{\infty} r^n = 1 + \sum_{m=0}^{\infty} r^{m+1}$$

let $m = n-1$
$n = m+1$

$$= 1 + r \sum_{n=0}^{\infty} r^n$$

$$\sum_{n=0}^{\infty} r^n \left[1 - r\right] = 1 \quad \text{so} \quad \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{for } -1 < r < 1$$

Another useful identity is

$$y[n] = \sum_{m=0}^{n} a^m = \frac{1-a^{n+1}}{1-a}$$

$$\sum_{m=0}^{\infty} a^m = \sum_{m=0}^{n} a^m + \sum_{m=n+1}^{\infty} a^m = \sum_{m=0}^{n} a^m + \sum_{k=0}^{\infty} a^{k+(n+1)}$$

$k = m - (n+1)$

$m = k + (n+1)$

$$= \sum_{m=0}^{n} a^m + a^{n+1} \sum_{m=0}^{\infty} a^m$$

$$\left(\sum_{m=0}^{\infty} a^m\right)(1-a^{n+1}) = \sum_{m=0}^{n} a^m = \frac{1}{1-a}(1-a^{n+1})$$

DT Convolution

$$y[n] = \sum_m x[n-m]\,h[m] = \sum_m x[m]\,h[n-m]$$

Ex: $h[n] = a^n u[n]$ for $|a| < 1$

let $x[n] = u[n]$

$y[n] = ?$

$$y[0] = \sum_m x[-m]\,h[m] = \sum_m x[-m]\,a^m u[m]$$

$$= \sum_m u[-m]\,a^m u[m] = 1$$

$$y[1] = \sum_m u[1-m]\,a^m u[m] = 1 + a$$

$$y[2] = \sum_m u[2-m]\,a^m u[m] = 1 + a + a^2$$

$$y[n] = 1 + a + a^2 + \cdots + a^n = \sum_{m=0}^{n} a^m = \frac{1-a^{n+1}}{1-a}$$

and for $n < 0$ $y[n] = 0$

Commutative

$$x[n] * h[n] = h[n] * x[n]$$

Associative

$$(x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n])$$

Distributive

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

Linearity

let $x[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$

$$y[n] = \sum_m x[m] h[n-m] = \sum_m (\alpha_1 x_1[m] + \alpha_2 x_2[m]) h[n-m]$$

$$= \alpha_1 \sum_m x_1[m] h[n-m] + \alpha_2 \sum_m x_2[m] h[n-m]$$

Time-invariant let $x[n] = x_1[n-k]$

$$y_1[n] = \sum_m x_1[n-k-m] h[m] = y[n-k]$$

Causal LTI System

$$y[n_0] = \sum_k h[k]\,x[n_0-k]$$

$$= \sum_{k=0}^{\infty} h[k]\,x[n_0-k] + \sum_{k=-\infty}^{-1} h[k]\,x[n_0-k]$$

$$= \overbrace{\left(h[0]\,x[n_0] + h[1]\,x[n_0-1] \cdots\right)}^{\text{present + past}} + \underbrace{\left(h[-1]\,x[n_0+1] + h[-2]\,x[n_0+2] \cdots\right)}_{\text{future}}$$

so let $h[n]=0$ for $n<0$ to achieve causality

so $$y[n] = \sum_{k=0}^{\infty} h[k]\,x[n-k] = \sum_{k=-\infty}^{n} x[k]\,h[n-k]$$