Review of Probability

We need probability and stochastic analysis to analyze and optimize communication systems.

Recall that random variables are described by probability measures.

1. **Probability Density Function (pdf)**
   \[ f_X(x) = \frac{1}{2} \frac{dF_X(x)}{dx} \]

2. **Cumulative Distribution Function (cdf)**
   \[ F_X(x) = \int_{-\infty}^{x} f_X(x) \, dx \]
Properties

1. \[ 0 \leq f_X(x) \leq 1 \]

2. \[ 1 = \int_{-\infty}^{\infty} f_X(x) \, dx \]

Example: Uniform distribution

\[ f_X(x) = U(a, b) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \]
Gaussian PDF
\[ f(x) = N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2}} \]

where \( \mu = \text{mean} \), \( \sigma = \text{standard deviation} \)
\( \sigma^2 = \text{variance} \)

\[ \text{Expected Value} \]
\[ E \left[ g(X) \right] = \int_{-\infty}^{\infty} g(x) f_x(x) \, dx \]

Moments
\[ E \left[ X^n \right] = \int_{-\infty}^{\infty} x^n f_x(x) \, dx \]
Example. First 2 moments of $U(0,1)$

$$E \{ x^3 \} = \int_{-\infty}^{\infty} x^3 f_X(x) \, dx = \int_{0}^{1} x^3 \, dx = \frac{x^4}{4} \bigg|_0^1 = \frac{1}{4}$$

$$E \{ x^2 \} = \int_{-\infty}^{\infty} x^2 f_X(x) \, dx = \int_{0}^{1} x^2 \, dx = \frac{x^3}{3} \bigg|_0^1 = \frac{1}{3}$$

Variance $\sigma^2 = E \{ x^2 \} - E \{ x \}^2 = E \{ x^2 \} - \mu^2$

$$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

Joint r.v.'s have joint, conditional and marginal statistics.

Joint pdf: $f_{X,Y}(x,y)$, and cdf $F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(x',y') \, dx' \, dy'$

Joint $E \{ Z \}$: $E \{ Z \} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z \, g(x,y) \, f_{X,Y}(x,y) \, dx \, dy$
Marginal statistics

\[ f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dy \]

and \( f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx \)

Conditional statistics

\[ f_{X|Y}(x|y) = f_{X,Y}(x,y) / f_Y(y) \]

\[ E_X g(x) | Y = y = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y) \, dx \]

\[ E_X g(x) | Y = y = \int_{-\infty}^{\infty} g(x) f_{X|Y}(y|x) \, dy \]

Statistical independence

\[ f_{X,Y}(x,y) = f_X(x) f_Y(y) \]

\[ E \{ X^3 Y^3 \} = E \{ X^3 \} E \{ Y^3 \} \]
Random Processes (r.p.)

Concept of the Ensemble Statistics

Given a r.p. \( \tilde{x}(t) \)

Sample functions

\( X_1(t) \)
\( X_2(t) \)
\( X_3(t) \)
\( X_4(t) \)
\( X_5(t) \)

Analogous to the "seed" value in a pseudo-random generator

\( t_1 \)
\( t_2 \)
What do r.p. have that r.v. don't?

A relationship between $t_1$ and $t_2$

We use the auto correlation function to characterize r.p. $\sim t_1$. $t_2$.

\[ R_{xx}(t_1, t_2) = E \left\{ \tilde{x}(t_1) \tilde{x}(t_2) \right\} \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2; t_1, t_2) \, dx_1 \, dx_2 \]

If the r.p. is stationary, in its first two moments, then we can write

\[ R_{xx}(t_1, t_2) = R_{xx}(t, t+\tau) = R_{xx}(\tau) \]

Terms like Wide Sense, Stationary (WSS)}
If \( \omega \leq \delta \), i.e. \( R_{xx}(\tau) = E \hat{x}(t) \hat{x}(t+\tau)^2 \) \( \forall t \)

then we know the Power Spectral Density is

\[
\begin{align*}
R_{xx}(\tau) & \xrightarrow{t} S_{xx}(f) \\
\text{Wiener - Khinchine relation}
\end{align*}
\]

Ex: **White Gaussian Noise**

Assume the stationary case s.t.

\( \tilde{\omega}(t) \sim N(0, \sigma^2) \equiv 0 \) mean **White Gaussian Noise**

\[
\begin{align*}
E \{ \tilde{\omega}(t)^3 \} &= 0 \\
E \{ \tilde{\omega}(t)^2 \}^3 &= \text{Var} \{ \tilde{\omega}(t)^2 \} = 0^2 \quad t = t
\end{align*}
\]
Auto correlation of WGN is

\[ R_{ww}(\tau) = E \{ \tilde{\omega}(t) \tilde{\omega}(t+\tau) \} = \delta(\tau) \sigma^2 \]

PSD = ?

\[ \int R_{ww}(\tau) \, d\tau = \sigma^2 \quad \text{white} \]

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**Ergodic Processes**

The practical value of ensemble statistics comes from the property of ergodicity. That is, the ensemble states approximate the time statistics if they are ergodic.

**Ex: Mean Ergodic**

\[ E \{ \tilde{X}(t) \}^2 = \int_{-\infty}^{\infty} \tilde{X}(x) \tilde{P}_{\tilde{X}(t)}(x; t) \, dx \]
\[ E \mathbf{R}_X(t, \tau) \mathbf{R}_X(t, \tau)^T = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \mathbf{R}_X(t, \tau) \mathbf{R}_X(t, \tau)^T dt \]

In general, an ergodic process has the property

\[ E \mathbf{R}_X(t, \tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \mathbf{R}_X(t, \tau) dt \]