EE511 lecture 9: Discrete Fourier Transform (DFT)

$$\underline{X}[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi k n}{N}} \quad \text{for } k = 0, 1, \cdots (N-1)$$

Inverse

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} \underline{X}[k]\, e^{j\frac{2\pi k n}{N}} \quad \text{for } n = 0, 1, \cdots (N-1)$$

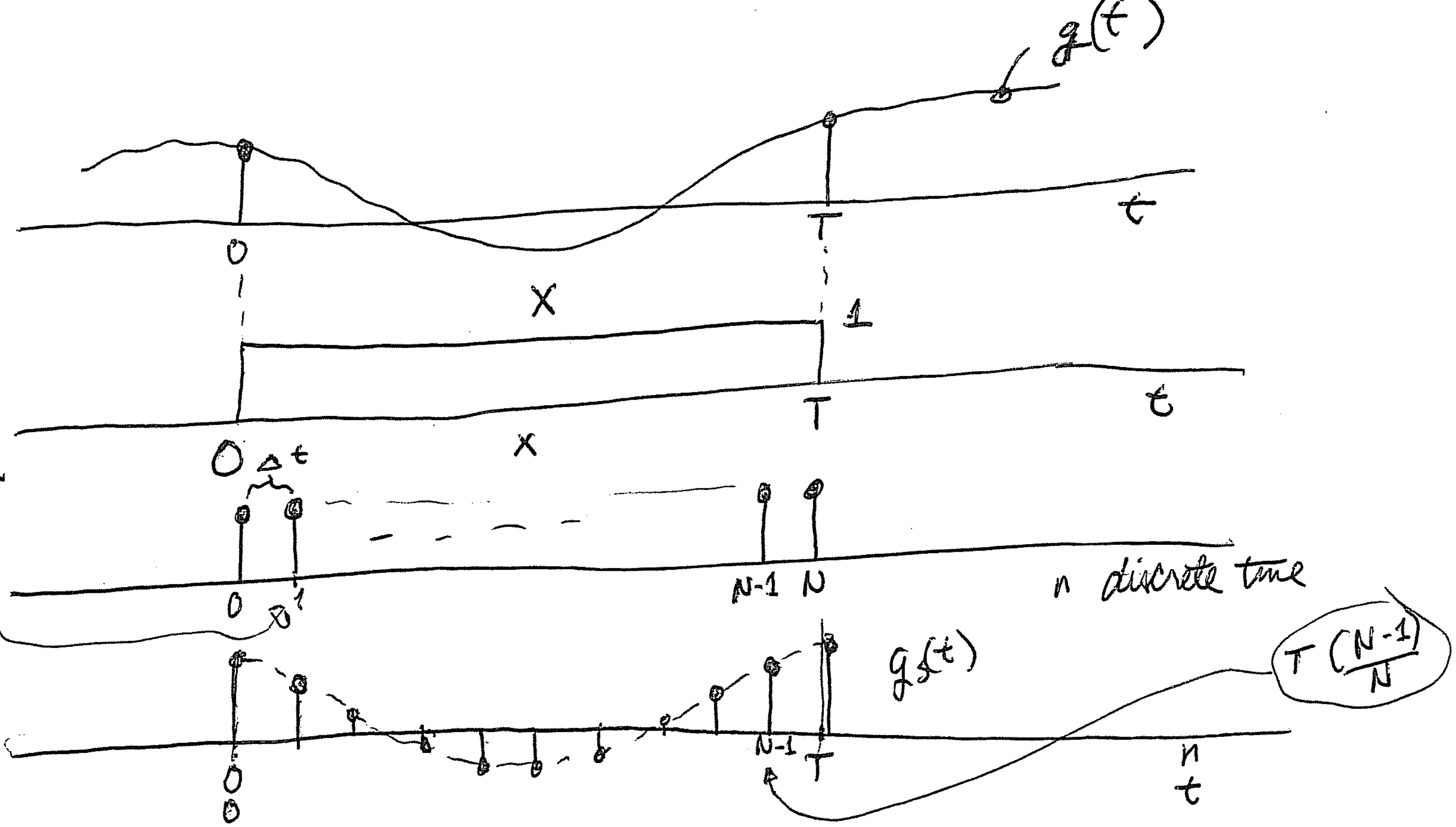
Theory

Consider

Window

Impulse Sample

$T_s = \Delta t = T/N$



Let's describe this sampling process mathematically

Let $\quad g_w(t) = g(t)\,\mathrm{rect}\left(\frac{t - T/2}{T}\right)$

then $\quad g_s(t) = g_w(t) \underbrace{\sum_n \delta\left(t - n\frac{T}{N}\right)}_{p(t)}$ $\quad \frac{T}{N} = \Delta t$

$$\mathcal{F}\{g_s(t)\} = \int_{-\infty}^{\infty} g(t)\,\mathrm{rect}\left(\frac{t - T/2}{T}\right) \sum_n \delta\left(t - \frac{nT}{N}\right) e^{-j2\pi f t}\,dt$$

$$= \int_0^T g(t) \sum_n \delta\left(t - n\frac{T}{N}\right) e^{-j2\pi f t}\,dt$$

$$= \int_0^T \sum_n \underbrace{g(t)}_{g[n] = g\left(n\frac{T}{N}\right)} \delta\left(t - n\frac{T}{N}\right) e^{-j2\pi f t}\,dt$$

$$\mathcal{F}\{g_s(t)\} = \sum_n g\left(\tfrac{nT}{N}\right) \int_0^T \delta\left(t - \tfrac{nT}{N}\right) e^{-j2\pi f t}\, dt$$

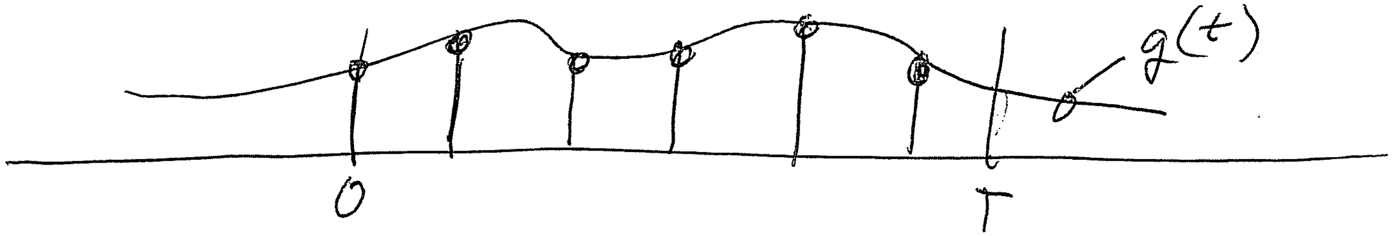
$$G_s(f) = \sum_{n=0}^{N-1} g\left(\tfrac{nT}{N}\right) e^{-j 2\pi f nT/N}$$

let $g[n] = g(nT/N)$ and $f = k/T$ be samples in the frequency domain s.t.

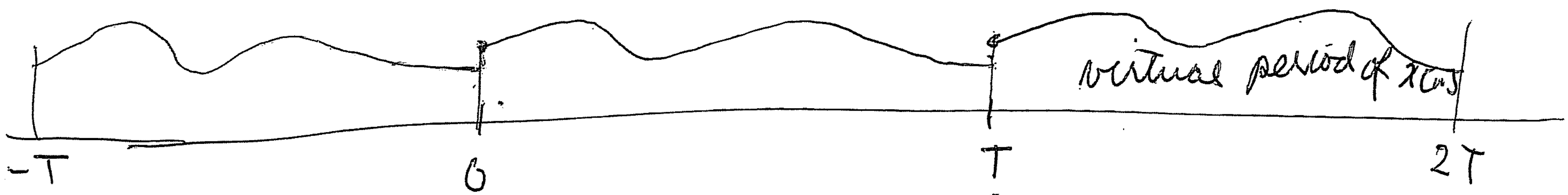
$$G_s\left(\tfrac{k}{T}\right) = \sum_{n=0}^{N-1} g[n]\, e^{-j\frac{2\pi kn}{N}} = \underbrace{G[k]}_{\text{DFT of } g[n]}$$

Using duality, sampling in the frequency domain infers the inverse Fourier Transform will yield a periodic time domain function.

For example



When ~~windowed~~ windowed and sampled to Discrete Time (DT), $g(t)$ will act like



$x[0] = x[N]$

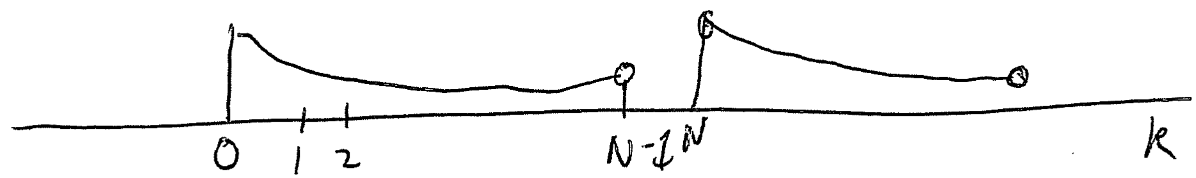
Note: "circular shift"

$x[n] = x[n + mN]$

or $x[n] = x[n \pm N]$

Properties of the frequency domain (5)-09

DFT frequency domain



circular ~~of~~ shift property $\underline{X}[k] = \underline{X}[k+N]$

If $x[n]$ is real then

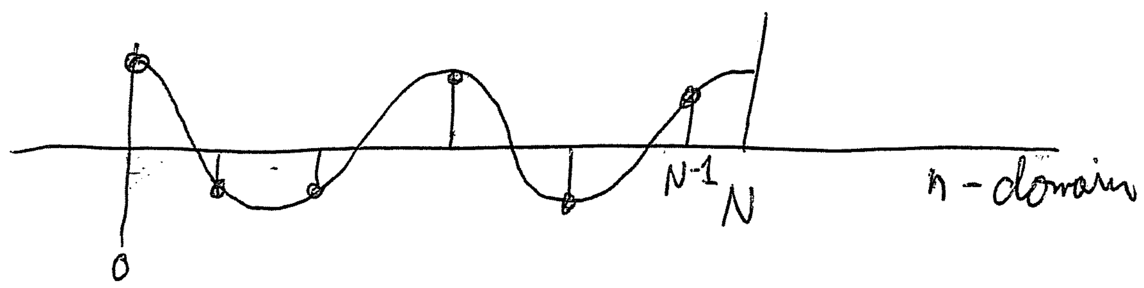
$$\text{Real}\{\underline{X}[k]\} = \text{Real}\{\underline{X}[N-k]\}$$ "symmetric" analogous to "even" function

$$\text{Imag}\{\underline{X}[k]\} = -\text{Imag}\{\underline{X}[N-k]\}$$ "anti-symmetric" analogous to "odd" function

If $x[n]$ is real and even $\{$ie, $x[n] = x[N-n]\}$

then $\underline{X}[k]$ is also real and even.

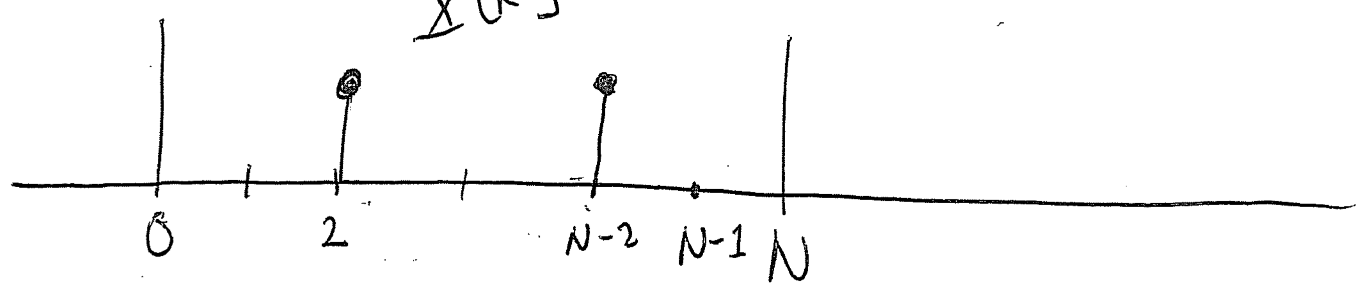
How do cosines map to k-domain? 6-09



count the number of cycles

for k cycles

$X[k]$



cosine is real and even if $x_c(n) = \cos\left(\frac{2\pi k_c n}{N}\right)$

$$X_c[k] = \frac{\delta[k-k_0] + \delta[k-(N-k_{k_0})]}{2}$$

sine waves are real and odd

$$X_s[k] = \frac{\delta[k-k_0] - \delta[k-(N-k_0)]}{2j}$$ where $x_s(n) = \sin\left(\frac{2\pi k_0}{N}\right)$