Lecture 7: Linear Systems

EE 5#1

$$x(t) \longrightarrow \boxed{h(t) \leftrightarrow H(f)} \longrightarrow y(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

$$Y(f) = X(f)\, H(f) \quad \text{where} \quad H(f) = \frac{Y(f)}{X(f)} \equiv$$ transfer function of filter

$h(t) \equiv$ impulse response of filter

let $x(t) = \delta(t)$

where $$y(t) = \int_{-\infty}^{\infty} \delta(\lambda)\, h(t-\lambda)\, d\lambda$$

$$= h(t)$$

Linearity and superposition

Let $g(x(t)) = y(t)$

$$y(t) = g(\alpha_1 x_1(t) + \alpha_2 x_2(t))$$

If $g(\cdot)$ is linear then

$$y(t) = \alpha_1 g(x_1(t)) + \alpha_2 g(x_2(t))$$
$$= y_1(t) + y_2(t)$$

Stability

A linear system is bounded input, bounded output (BIBO) ~~of~~ stable if every bounded input results in a bounded output.

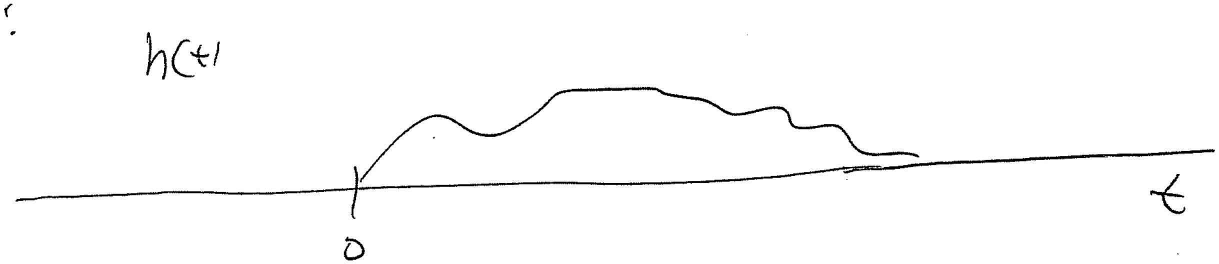
We have BIBO stability if $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

Causality

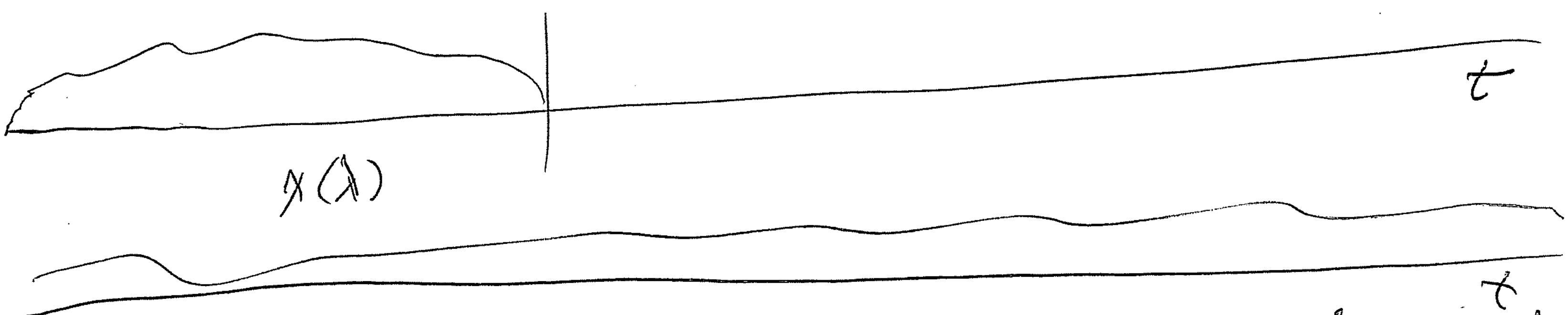
A system is causal if it does not anticipate the input. For time-invariant causal system

$$h(t) = 0, \quad t < 0$$

NOTE:



in conv $h(t-\lambda)$



refer to text page 59 for Wiener and Paley theorem.

Symmetry Properties of $H(f)$

Let $H(f) = |H(f)| \exp\left[j \angle H(f)\right]$

For a real-time function $h(t)$

$$|H(f)| = |H(-f)| \quad \text{(even)}$$

and $$\angle H(f) = -\angle H(-f) \quad \text{(odd)}$$

Power Transfer Function

Consider an energy signal (finite length) $y(t)$ as a response $y(t) = x(t) * h(t)$

The auto-correlation is $R_{yy}(\tau) = \int_{-\infty}^{\infty} y(\lambda)\, y^*(\lambda-\tau)\, d\lambda$

The Power Spectral Density is

$$P_{yy}(f) = \mathcal{F}\{R_{yy}(\tau)\} = Y(f)\,Y^*(f) = |Y(f)|^2$$

$$P_{yy}(f) = X(f)\,H(f)\,X^*(f)\,H^*(f) = |X(f)|^2\,|H(f)|^2$$

$$= P_{xx}(f)\,|H(f)|^2$$

Power Transfer Function $= |H(f)|^2 = \dfrac{P_{yy}(f)}{P_{xx}(f)}$

Finding Power Transfer of a Power Signal

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} y^2(t)\,dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} y_T^2(t)\,dt$$

Using Parseval's Theorem

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |Y_T(f)|^2\,df$$

$$P = \int_{-\infty}^{\infty} \underbrace{\lim_{T\to\infty} \frac{|Y_T(f)|^2}{T}}_{\text{PSD of } y \text{ or } P_{yy}(f)} df$$

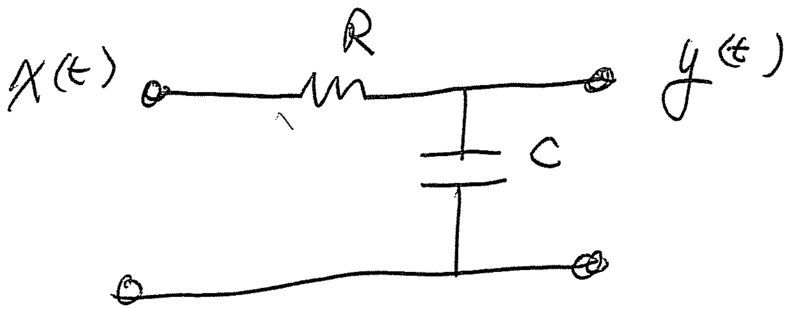
$$P_{yy}(f) = \lim_{T\to\infty} \frac{|Y_T(f)|^2}{T} \quad \text{and} \quad |Y_T(f)|^2 = |X_T(f)|^2 |H(f)|^2$$

$$P_{yy}(f) = |H(f)|^2 \underbrace{\lim_{T\to\infty} \frac{|X_T(f)|^2}{T}}_{P_{xx}(f)}$$

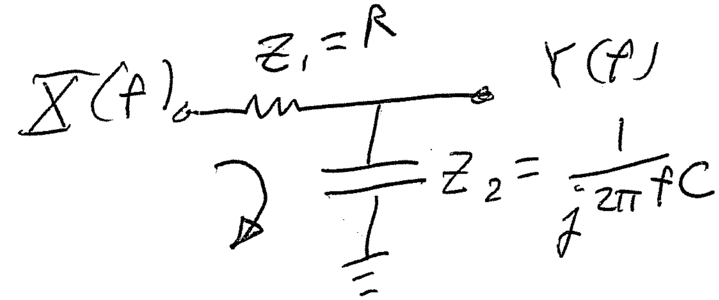
$\therefore$ Power Transfer Function

$$|H(f)|^2 = \frac{P_{yy}(f)}{P_{xx}(f)}$$

Ex: PTF of a simple RC circuit



$\Rightarrow$



$$H(f) = \frac{Y(f)}{X(f)} = \frac{Z_2}{Z_1 + Z_2} = \frac{\frac{1}{j2\pi fC}}{R + \frac{1}{j2\pi fC}} = \frac{1}{1 + j2\pi fRC}$$

Power Transfer function

$$|H(f)|^2 = \frac{1}{1 + (2\pi fRC)^2}$$

Group and Phase Delay

Group delay is defined by

$$T_g(f) = -\frac{1}{2\pi}\frac{d\theta(f)}{df}$$

where we have $Y(f) = |Y(f)|\, e^{+j\theta(f)}$