EE511

Lecture 6 : PSD and auto Correlation

Deterministic definition of Power Spectral Density (PSD)

given $w_T(t) = w(t)\,\mathrm{rect}\left(\frac{t}{T}\right)$

The average power is

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} w^2(t)\,dt = \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty} w_T^2(t)\,dt$$

By Parseval's rule $E = \int_{-\infty}^{\infty} |w_T(t)|^2\,dt = \int_{-\infty}^{\infty} |W_T(f)|^2\,df$

$$P = \lim_{T\to\infty} \langle w_T^2(t)\rangle = \int_{-\infty}^{\infty} P_w(f)\,df = \mathcal{F}^{-1}\{P_w(f)\}\Big|_{t=0}$$

$$P_w(f) \triangleq \lim_{T\to\infty}\left(\frac{|W_T(f)|^2}{T}\right) \triangleq \text{PSD}$$

Auto correlation Function

$$R_w(\tau) \triangleq \lim_{T\to\infty} \langle w(t)\,w(t+\tau)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} w(t)\,w(t+\tau)\,dt$$

Wiener - Khintchine Theorem

$$R_w(\tau) \xleftrightarrow{\mathcal{F}} P_w(f)$$

$$f(t) * f^*(-t) \xleftrightarrow{\mathcal{F}} |F(f)|^2 = F(f)\,F^*(f)$$

auto-correlation

Properties

$$P = \langle w^2(t) \rangle = W_{rms}^2 = \int_{-\infty}^{\infty} P_w(f)\,df = R_w(0)$$

Ex: Auto - correlation of $A \sin \omega_0 t$

$$R_w(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} A^2 \underbrace{\sin \omega_0 t \sin \omega_0 (t+\tau)}_{\frac{\cos \omega_0 \tau - \cos(2\omega_0 t + \tau)}{2}} dt$$

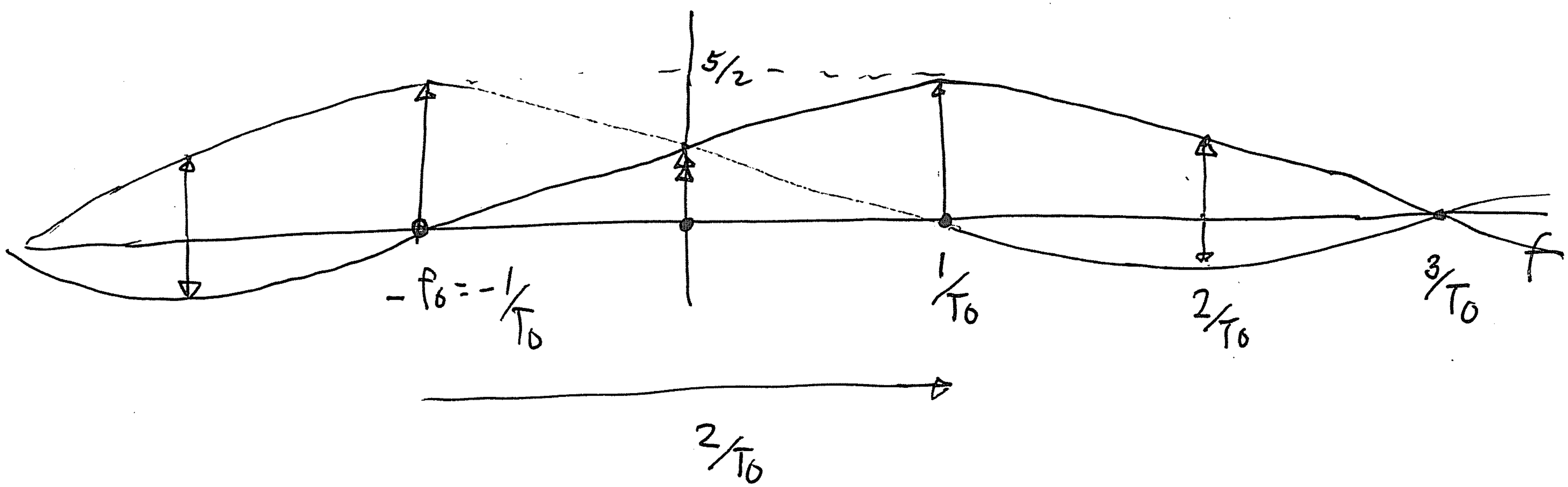
$$= \lim_{T \to \infty} \frac{1}{2T} \left[A^2 \cos \omega_0 \tau \int_{-T/2}^{T/2} dt - A^2 \underbrace{\int_{-T/2}^{T/2} \cos(2\omega_0 t + \tau)\,dt}_{0} \right]$$

$$R_w(\tau) = \frac{A^2}{2} \cos \omega_0 \tau$$

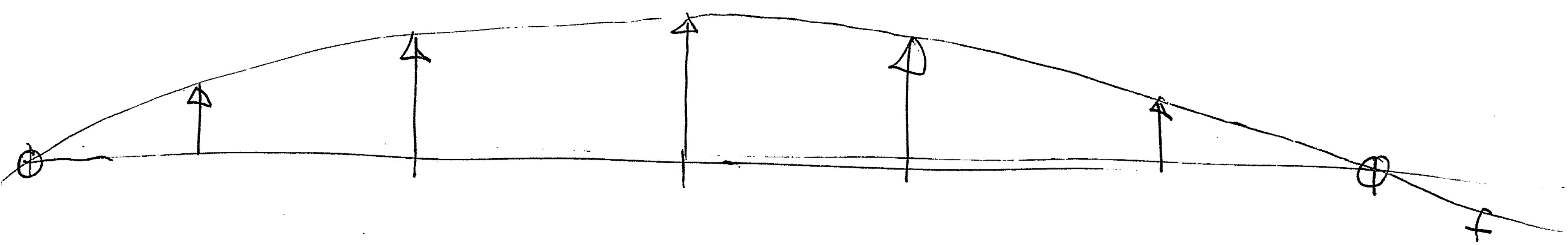
To graph this, we need to find nulls

(3)

$$S_a\left(\pi \frac{f T_0}{2}\right) \Rightarrow \frac{\pi f T_0}{2} = \pi n \Rightarrow f = \frac{2n}{T_0} \quad \text{even harmonics}$$



$$PSD = \sum_n |C_n|^2 \delta(f - n f_0)$$



$$C_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} v_o(t)\, e^{-j 2\pi f_0 n t}\, dt$$

$$v_o(t) = 10 \operatorname{rect}\left(\frac{t}{T_0/2}\right) \cos 2\pi f_0 t$$

$$C_n = \frac{1}{T_0} \int_{-\infty}^{\infty} 10 \operatorname{rect}\left(\frac{t}{T_0/2}\right) \cos 2\pi f_0 t \; e^{-j 2\pi f t}\, dt \Big|_{f = f_0 n}$$

Using our FT (rect) trick we know

$$\operatorname{rect}\left(\frac{t}{T_0}\right) \operatorname{rect}\left(\frac{t}{T_0/2}\right) = \operatorname{rect}\left(\frac{t}{T_0/2}\right)$$

We can use the FT to get C_n

$$C_n = \frac{10}{T_0} \frac{T_0}{2} Sa\left(\pi f \frac{T_0}{2}\right) * \frac{\delta(f-f_0) + \delta(f+f_0)}{2} \Big|_{f = f_0 n}$$

$$= \frac{5}{2} \left[Sa\left(\pi \frac{(f-f_0) T_0}{2}\right) + Sa\left(\pi \frac{(f+f_0) T_0}{2}\right) \right] \Big|_{f = f_0 n}$$

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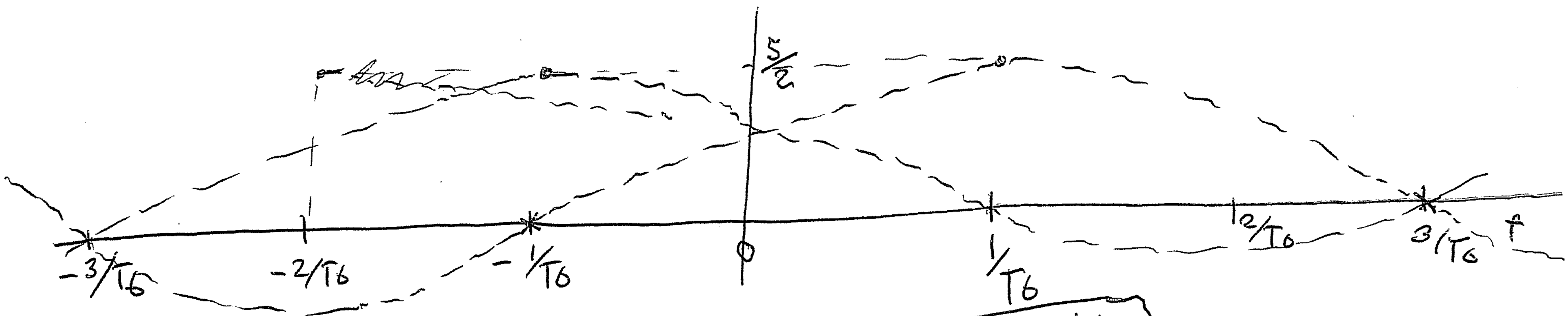
$$C_n = \frac{5}{2}\left[Sa\left(\pi \frac{(f-f_0)T_0}{2}\right) + Sa\left(\pi \frac{(f+f_0)T_0}{2}\right)\right]_{f=f_0 n}$$

To graph this, we need to find nulls

$$Sa\left(\pi \frac{f T_0}{2}\right) \Rightarrow \pi \frac{f T_0}{2} = \pi n \Rightarrow f = \frac{2n}{T_0} \Big\}\ \text{even harmonics}$$



$\boxed{f_0 = 1/T_0}$

$$PSD = \sum_n |C_n|^2\, \delta(f - n f_0)$$

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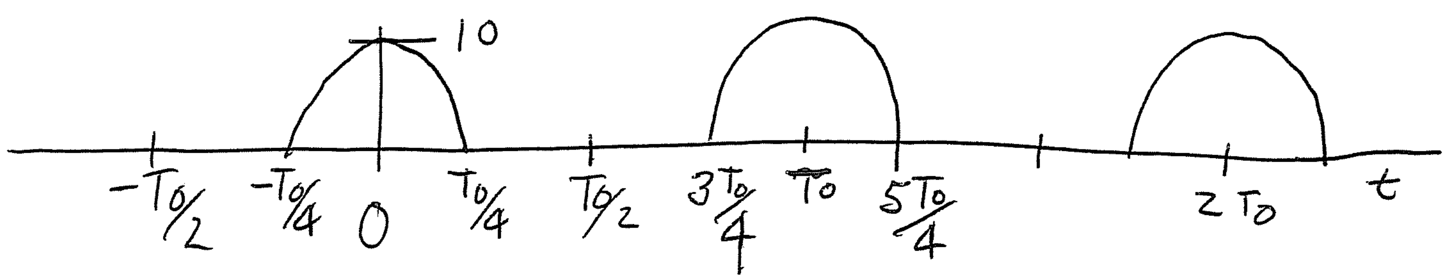

assume $\omega_0 = 2\pi f_0$, $T_0 = 1/f_0$

$$C_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} v_0(t)\, e^{-j 2\pi f_0 n t}\, dt$$

$$v_0(t) = 10 \,\text{rect}\left(\frac{t}{T_0/2}\right) \cos 2\pi f_0 t$$

$$C_n = \frac{1}{T_0} \int_{-\infty}^{\infty} 10 \,\text{rect}\left(\frac{t}{T_0/2}\right) \cos 2\pi f_0 t \; e^{-j 2\pi f t}\, dt \Bigg|_{f = f_0 n}$$

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$$C_n = \frac{10}{T_0} \frac{T_0}{2} Sa\left(\pi \frac{f T_0}{2}\right) * \frac{\delta(f - f_0) + \delta(f + f_0)}{2} \Bigg|_{f = f_0 n}$$

$$P_w(f) = \mathcal{F}\{R_w(\tau)\} = \mathcal{F}\{A^2/_2 \cos \omega_0 \tau\}$$

$$= \frac{A^2}{2} \, \frac{\delta(\omega-\omega_0) + \delta(\omega+\omega_0)}{2}$$

$$P = \int_{-\infty}^{\infty} P_w(f)\, df = \frac{A^2}{4} + A^2/_4 = A^2/_2 = R_w(0)$$

Finding PSD from C_n of a FS

Use the identity $P(f) = \sum_n |C_n|^2 \delta(f - nf_0)$

Ex: Find the PSD of

$$v_n(t) = \begin{cases} 10 \cos \omega_0 t & |t - nT_0| < T_0/4 \\ 0 & \text{elsewhere} \end{cases}$$