Lecture 5: EE511 Fourier Series (FS) expansion

Recall: FS epansion

Given $s(t) = s(t+T)$ is a periodic function

then $s(t) = \sum_n C_n e^{j 2\pi f_0 n t}$ where $f_0 = \frac{1}{T}$

and $C_n = \frac{1}{T} \int_{-T/2}^{T/2} s(t) e^{-j 2\pi f_0 n t} dt$

Note: given $s(t)$ then

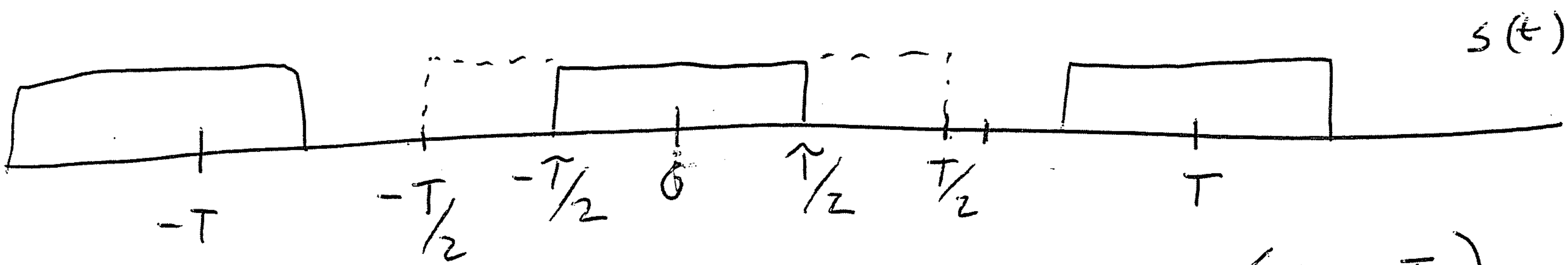
$$C_n = \frac{1}{T} \mathcal{F} \left\{ \mathrm{rect}\left(t/T\right) s(t) \right\} \Big|_{f=f_0 n} = \frac{1}{T} T Sa(\pi f T) * S(f) \Big|_{f=f_0 n}$$

$$C_n = Sa(\pi f_0 n T) * S(f_0 n)$$

The FS coefficients are sinc functions convolved with the spectra of the waveform.

FS of a square wave

Consider the periodic part such that



In this case $s(t) = \sum_n \text{rect}\left(\frac{t - nT}{\tau}\right)$

$$C_n = \frac{1}{T} \mathcal{F}\left\{ \text{rect}\left(t/T\right) s(t) \right\} \Big|_{f = f_0 n}$$

$$= \frac{1}{T} \int_{-\infty}^{\infty} \text{rect}\left(t/T\right) \text{rect}\left(\frac{t - \overset{n=0}{nT}}{\tau}\right) e^{-j 2\pi f t} dt \Big|_{f = f_0 n}$$

$$C_n = \frac{1}{T} \int_{-\infty}^{\infty} \text{rect}\left(t/T\right) \text{rect}\left(\frac{t}{\tau}\right) e^{-j 2\pi f t} dt \Big|_{f = f_0 n}$$

$$C_n = \frac{1}{T} \int_{-\infty}^{\infty} \underbrace{\mathrm{rect}(t/\tau)}_{\text{assume } \tau \le T} e^{-j 2\pi f t}\, dt \Big|_{f = f_0 n}$$

$$= \tau/T \; Sa(\pi f \tau) \Big|_{f = f_0 n = \frac{n}{T}} = \tau/T \; Sa\left(\pi n \frac{\tau}{T}\right)$$

so if

$$s(t) = \sum_n C_n e^{j 2\pi f_0 n t}$$

$$\boxed{s(t) = \sum_n (\tau/T)\, Sa(\pi n \tau/T)\, e^{j \frac{2\pi n t}{T}}}$$

infinite sum of sinc functions

How does this compare to the FT of the squarewave?

Recall $s(t) = \mathrm{rect}(t/\tau) * \sum_n \delta(t - nT)$

$$S(f) = \tau/T \; Sa(\pi f \tau) \sum_k \delta(f - k/T)$$

$$S(f) = \sum_k \left[\left(\tau/T\right) Sa(\pi f \tau) \right] \delta(f - k/T)$$

$$= \sum_k \underbrace{\left[\left(\tau/T\right) Sa\left(\pi \frac{k\tau}{T}\right) \right]}_{C_k \,\equiv\, \text{FS coefficient}} \delta(f - k/T)$$

Then

$$s(t) = \mathcal{F}^{-1}\{S(f)\}$$

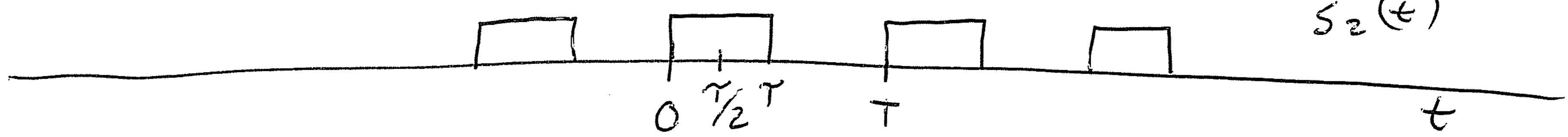
$$= \int_{-\infty}^{\infty} \sum_k \left[\left(\tau/T\right) Sa\left(\pi \frac{k\tau}{T}\right) \right] \delta(f - k/T)\, e^{j2\pi f t}\, df$$

$$= \sum_k \left[\left(\tau/T\right) Sa\left(\frac{\pi k \tau}{T}\right) \right] \int_{-\infty}^{\infty} \delta(f - k/T)\, e^{j2\pi f t}\, df$$

samples @ $f = k/T$

$$= \sum_k \left[\left(\tau/T\right) Sa\left(\frac{\pi k \tau}{T}\right) \right] e^{j\frac{2\pi k t}{T}} = \text{FS of } s(t)$$

Example: 50% duty cycle square wave and align the $s(t)$ to $s_2(t)$



50% duty yields $\tau/T = 0.5$ so $\tau = T/2$

what about the time shift $s_2(t) = s(t - T/4)$

so $s_2(t) = \text{rect}\left(\frac{t - T/4}{\tau}\right) * \sum_n \delta(t - T/4 - nT)$

$$S_2(f) = e^{-j2\pi \frac{T}{4} f} S(f)$$

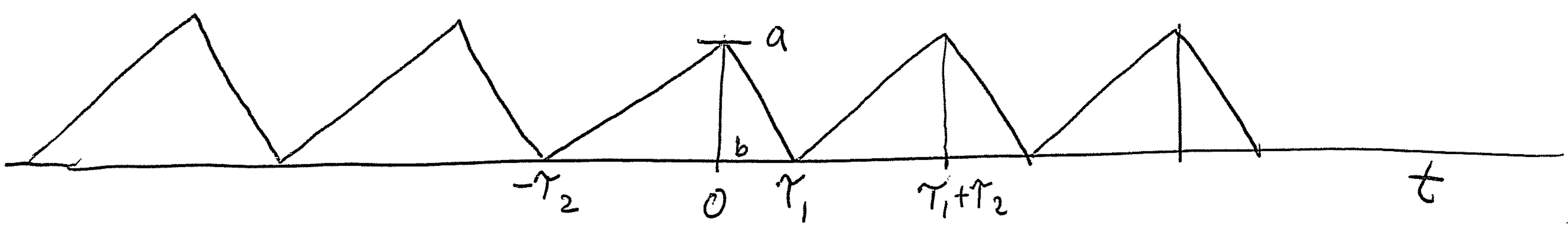
$$= e^{-j2\pi \frac{T}{4} f} \left(\tau/T\right) Sa(\pi f \tau) \sum_k \delta(f - k/T)$$

where $\tau = T/2$

Where is the FS prefered over the FT?

When the periodic waveform is complicated and/or the FT is not known

An example of a difficult function is the sawtooth waveform



$$C_n = \frac{1}{\tau_1+\tau_2}\Bigg[\underbrace{\int_{-\tau_2}^{0}(a_2 t + b_2)\, e^{-j2\pi f_0 n t}\, dt + \int_{0}^{\tau_1}(a_1 t + b_1)\, e^{-j2\pi f_0 n t}\, dt}_{\text{piece wise integration}}$$

We can find a_1, b_1 from $t=0$ $a=b_1$, $a=b_2$

@ $t=\tau_1$ $\quad 0 = a_1\tau_1 + b_1$ so $a_1 = -a/\tau_1$

@ $t=-\tau_2$ $\quad 0 = a_2(-\tau_2)+b_2$, $a_2 = a/\tau_2$

Break integrals up into 2 integrals each

$$\int_{-\tau_2}^{0} a_2 t\, e^{-j2\pi f_0 n t}\,dt + \int_{-\tau_2}^{0} b_2\, e^{-j2\pi f_0 n t}\,dt$$

Use Integral Table to obtain the anti-derivative and evaluate at the limits.