Lecture 3 Fourier Transform EE511

Projection Integral and the Fourier Transform (FT)

Recall the projection integral but remove the normalization.

i.e. $$y_{ab} = \cancel{\frac{1}{T}} \int_{t_1}^{t_2} a(t)\,b(t)\,dt \equiv a \text{ proj. onto } b$$

Consider the case where $b(t) = e^{-j2\pi f t}$

so $$y_{ab} = \int_{t_1}^{t_2} a(t)\,e^{-j2\pi f t}\,dt$$

The FT is the limit of the above s.t.

$$\mathcal{F}\{a(t)\} = \lim_{\substack{t_1 \to -\infty \\ t_2 \to \infty}} \int_{t_1}^{t_2} a(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} a(t)\, e^{-j2\pi f t}\, dt = A(f)$$

The FT is invertable s.t.

$$a(t) = \int_{-\infty}^{\infty} A(f)\, e^{j2\pi f t}\, df$$

Properties of the FT

(1) <u>Duality</u> $\mathcal{F}\{\mathcal{F}\{x(t)\}\} = x(-t)$

i.e., FT of a FT yields the original function.

2. Time Shift - invariance

Let $G(f) = \mathcal{F}\{x(t-\tau)\} = \int_{-\infty}^{\infty} x(t-\tau)e^{-j2\pi ft}\,dt$

Let $\lambda = t-\tau \qquad d\lambda = dt$

limits: $\infty \to \infty = \infty - \tau, \quad -\infty = -\infty - \tau$

$$\mathcal{F}\{x(t-\tau)\} = \int_{-\infty}^{\infty} x(\lambda)\, e^{-j2\pi f(\lambda+\tau)}\,d\lambda$$

$$= e^{-j2\pi f\tau} \int_{-\infty}^{\infty} x(\lambda)\, e^{-j2\pi f\lambda}\,d\lambda$$

$$G(f) = e^{-j2\pi f\tau}\, X(f)$$

Note: if the $G(f)$ is magnitude squared then $|G(f)|^2 = G(f)\,G^*(f)$

3 Frequency shift

$$g(t) = \mathcal{F}^{-1}\{W(f-f_0)\} = \int_{-\infty}^{\infty} W(f-f_0)\, e^{j2\pi f t}\, df$$

let $\lambda = f - f_0$ then

$$g(t) = \int_{-\infty}^{\infty} W(\lambda)\, e^{j2\pi(\lambda+f_0)t}\, d\lambda$$

$$= e^{j2\pi f_0 t}\, w(t)$$

4 Convolution

$$w_1(t) * w_2(t) \xleftrightarrow{\mathcal{F}} W_1(f)\, W_2(f)$$

5 correlation

$$w_1(t) \circ w_2(t) = w_1(t) * w_2^*(-t) \longleftrightarrow W_1(f)\, W_2^*(f)$$

Note : if $w_1(t) = w_2(t)$ then $w_1(t) \circ w_2(t) \longleftrightarrow |W_1(f)|^2$

6 Multiplication

$$w_1(t)\, w_2(t) \xleftrightarrow{\mathcal{F}} W_1(f) * W_2(f)$$

We use the convolution property to make numerically efficient approximations using the Fast Fourier Transform (FFT)

i.e.

$$y[n] = x_1[n] * x_2[n] = FFT^{-1}\left(FFT(x_1[n]) .* FFT(x_2[n])\right)$$

which approximates

$$y(t) = x_1(t) * x_2(t) = \mathcal{F}^{-1}\left\{X_1(f)\, X_2(f)\right\}$$

Proof of Convolution

Let's start with $\mathcal{F}^{-1}\{\ \}$

i.e. $$? = \int_{-\infty}^{\infty} W_1(f)\, W_2(f)\, e^{j2\pi ft}\, df$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \omega_1(\lambda)\, e^{-j2\pi f\lambda}\, d\lambda \right] \left[\int_{-\infty}^{\infty} \omega_2(\beta)\, e^{-j2\pi f\beta}\, d\beta \right] e^{j2\pi ft}\, df$$

$$= \iint_{-\infty}^{\infty} \omega_1(\lambda)\, \omega_2(\beta) \Bigg[\underbrace{\underbrace{\int_{-\infty}^{\infty} e^{-j2\pi f\lambda}\, e^{-j2\pi f\beta}\, e^{j2\pi ft}\, df}_{\int_{-\infty}^{\infty} e^{-j2\pi f(\lambda+\beta)}\, e^{j2\pi ft}\, df}}_{\delta(t-(\lambda+\beta))} \Bigg] d\lambda\, d\beta$$

$$= \iint_{-\infty}^{\infty} w_1(\lambda)\, w_2(\beta)\, \underbrace{\delta(t-(\lambda+\beta))}_{\delta((t-\beta)-\lambda)}\, d\lambda\, d\beta$$

samples when $\lambda = t-\beta$

$$= \int_{-\infty}^{\infty} w_1(t-\beta)\, w_2(\beta)\, d\beta$$

$$= \boxed{\int_{-\infty}^{\infty} w_2(\beta)\, w_1(t-\beta)\, d\beta = w_1(t) * w_2(t)}$$