Lecture 10 Discrete Fourier Transform (DFT) ①

$$\mathbb{X}[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi k n}{N}} \quad \text{for } k = 0, 1, \cdots (N-1)$$

Inverse

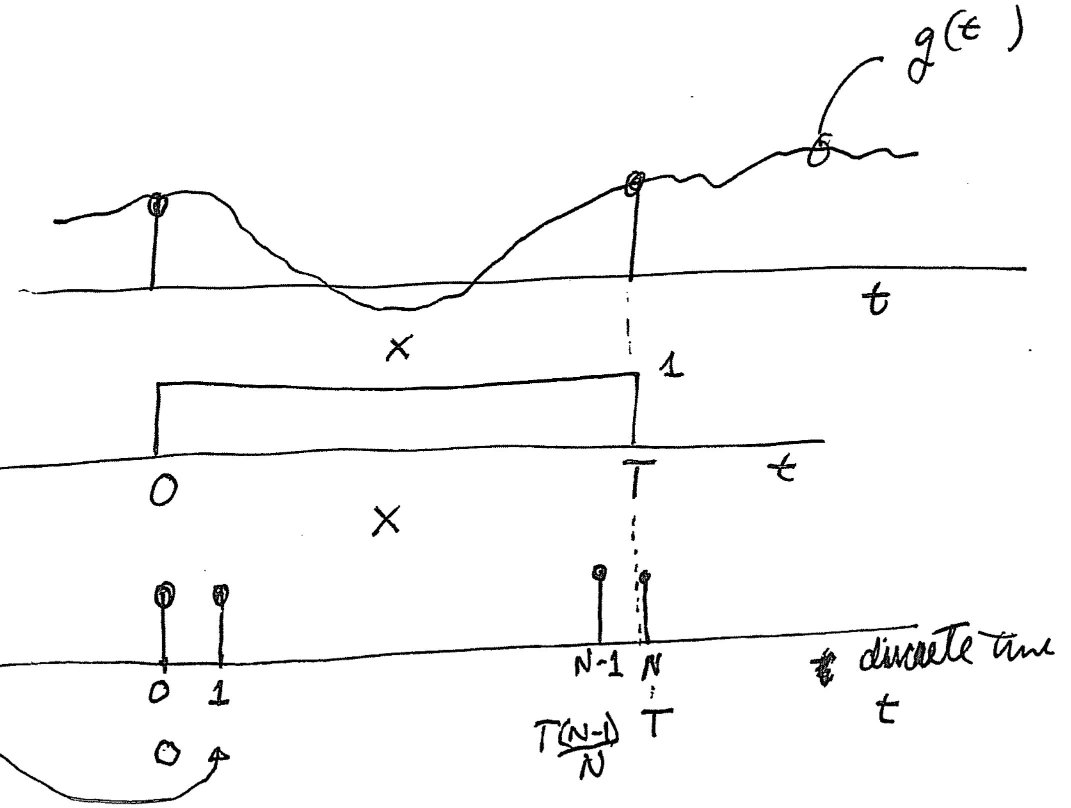
$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} \mathbb{X}[k]\, e^{j\frac{2\pi k n}{N}} \quad \text{for } n = 0, 1, \cdots (N-1)$$

Theory

Consider

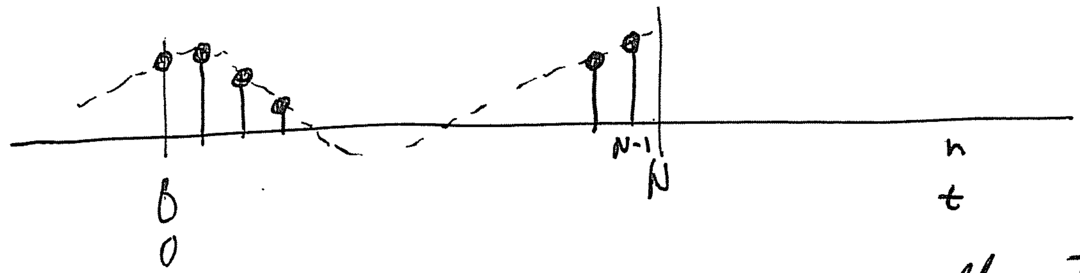
Window

Impulse Sample



$T_s = \Delta t = T/N$

$g_s(t)$



Let's describe this sampling process mathematically

Let $g_w(t) = g(t)\, \text{rect}\left(\frac{t - T/2}{T}\right)$

then $g_s(t) = g_w(t) \underbrace{\sum_n \delta\left(t - n\frac{T}{N}\right)}_{p(t)}$

$$\mathcal{F}\{g_s(t)\} = \int_{-\infty}^{\infty} g(t)\, \text{rect}\left(\frac{t - T/2}{T}\right) \sum_n \delta\left(t - n\frac{T}{N}\right) e^{-j2\pi f t}\, dt$$

$$= \int_0^T g(t) \sum_n \delta\left(t - n\frac{T}{N}\right) e^{-j2\pi f t}\, dt$$

$$= \int_0^T \sum_n \underbrace{g(t)}_{g[n] = g(nT/N)} \delta\left(t - \frac{nT}{N}\right) e^{-j2\pi f t}\, dt$$

$$\tilde{\mathcal{F}}\{g_s(t)\} = \sum_n g\left[n\tfrac{T}{N}\right] \int_0^T \delta\left(t - n\tfrac{T}{N}\right) e^{-j 2\pi f t}\, dt$$

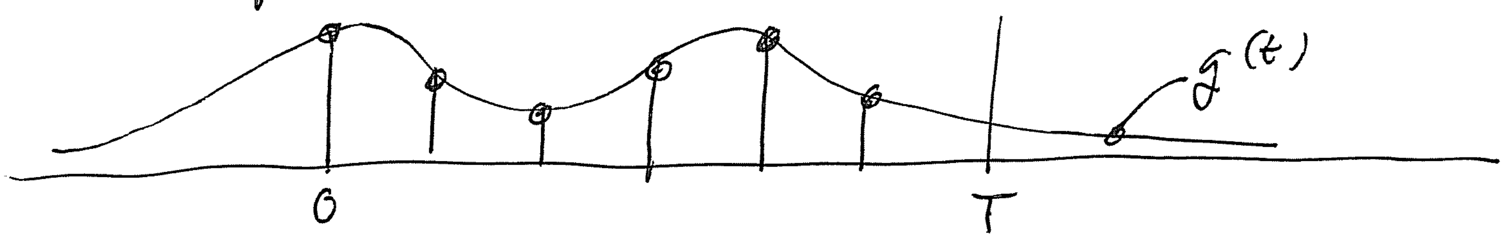
$$G_s(f) = \sum_{n=0}^{N-1} g\left(n\tfrac{T}{N}\right) e^{-j 2\pi f n \frac{T}{N}}$$

We let $g[n] = g(nT/N)$ and $f = k/T$ be samples in the frequency domain so

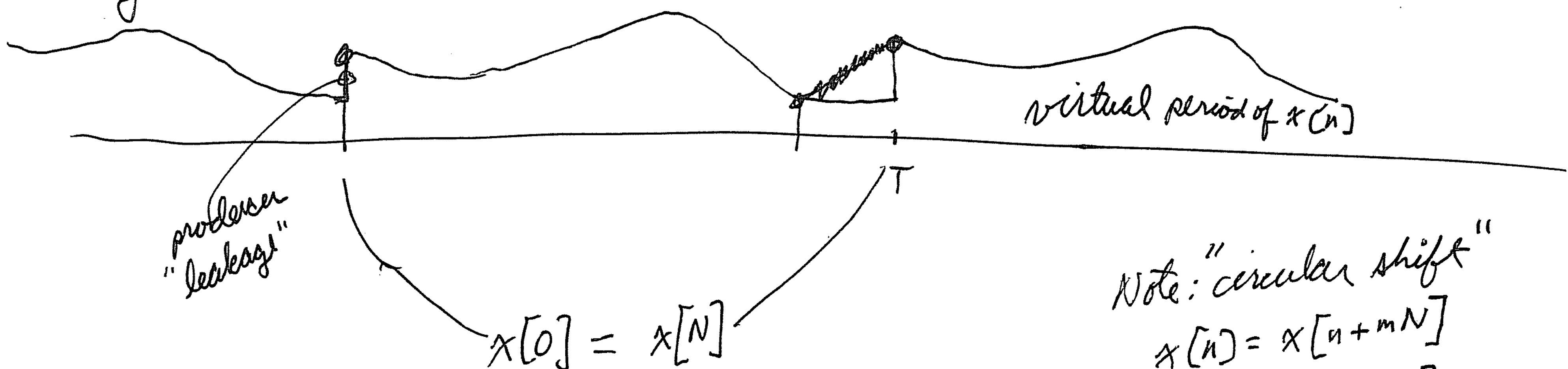
$$G_s\left(\tfrac{k}{T}\right) = \sum_{n=0}^{N-1} g[n]\, e^{-j \frac{2\pi k n}{N}} = \underbrace{G[k]}_{\text{DFT of } g[n]}$$

Using duality, sampling in the frequency domain ~~[illegible]~~ infers the inverse Fourier Transform will yield a periodic time domain function.

For example



When windowed and sampled to Discrete Time (DT), g(t) will act like

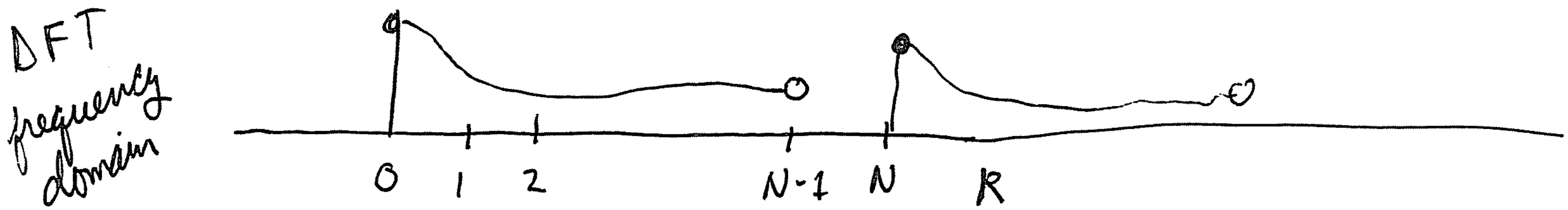


Note: "circular shift"

$x(n) = x[n+mN]$

or $x[n] = x[n \pm N]$

Properties of the frequency domain



Circular Shift property $X[k] = X[k+N]$

If $x[n]$ is real then

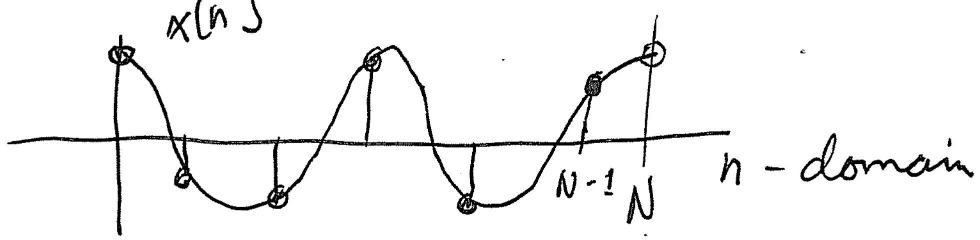
$$\text{Real}\{X[k]\} = \text{Real}\{X[N-k]\}$$ "symmetric" analogous to "even" function

$$\text{Imag}\{X[k]\} = -\text{Imag}\{X[N-k]\}$$ "anti-symmetric" analogous "odd" function

If $x[n]$ is real and even $\{x[n] = x[N-n]\}$

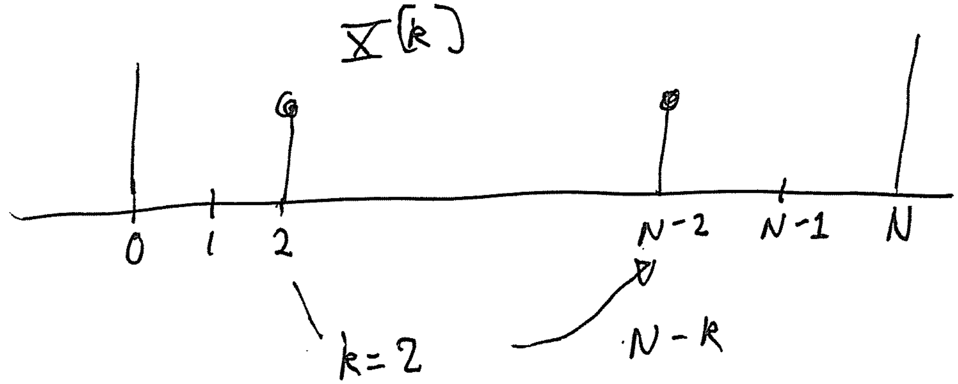
then $X[k]$ is also real and even.

How do cosines map to k-domain?



count the number of cycles

for k cycles



cosine is real and even if $x_c[n] = \cos\left(2\pi \frac{k_c n}{N}\right)$

$$X_c[k] = \frac{\delta[k-k_0] + \delta[k-(N-k_0)]}{2}$$

sine waves are real and odd

$$X_s[k] = \frac{\delta[k-k_0] - \delta[k-(N-k_0)]}{2j} \quad \text{where } x_s[n] = \sin\left(\frac{2\pi k_0 n}{N}\right)$$