Lecture 8 Fourier Series (FS) expansion

Recall: FS expansion

given $s(t) = s(t+T)$ is a periodic function

then $$s(t) = \sum_n c_n e^{j2\pi f_0 n t}$$

where $f_0 = 1/T$

and $$c_n = 1/T \int_{-T/2}^{T/2} s(t) e^{-j2\pi f_0 n t} dt$$

Note: given $s(t)$ then

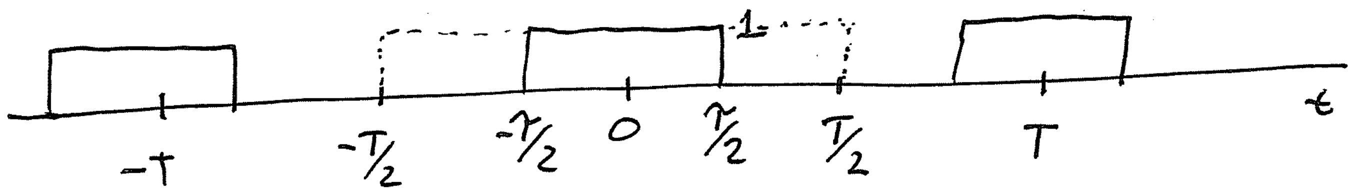
$$c_n = \frac{1}{T} \mathcal{F}\left\{ rect\left(\frac{t}{T}\right) s(t) \right\}\Big|_{f = f_0 n} = 1/T \; T\, Sa(\pi f T) * S(f) \Big|_{f = f_0 n}$$

$$c_n = Sa(\pi f_0 n T) * S(f_0 n)$$

The FS coefficients are sinc functions convolved with the spectra of the waveform

FS of a square wave

Consider the periodic part such that



In this case $s(t) = \sum_{n=-\infty}^{\infty} \mathrm{rect}\left(\frac{t-nT}{\tau}\right)$

$$c_n = \frac{1}{T} \mathcal{F}\left\{ \mathrm{rect}\left(\frac{t}{T}\right) s(t) \right\}\Big|_{f=f_0 n} = \frac{1}{T}\int_{-\infty}^{\infty} \mathrm{rect}\left(\frac{t}{T}\right)\mathrm{rect}\left(\frac{t}{\tau}\right) e^{-j2\pi f t}\,dt\Big|_{f=f_0 n}$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} \mathrm{rect}\left(t/\tau\right) e^{-j2\pi f t}\,dt\Big|_{f=f_0 n}$$

$$= \frac{\tau}{T} Sa(\pi f \tau)\Big|_{f=f_0 n = n/T} = \tau/T \; Sa\left(\pi n \frac{\tau}{T}\right)$$

$$s(t) = \sum_n C_n e^{j 2\pi f_0 n t}$$

$$\boxed{s(t) = \sum_n \left(\tau/T\right) Sa\left(\pi n \tau/T\right) e^{j \frac{2\pi n t}{T}}}$$

infinite sum of sinc functions

How does this compare to the FT of the square wave?

Recall $\quad s(t) = \text{rect}\left(t/\tau\right) * \sum_n \delta(t - nT)$

$$S(f) = \frac{\tau}{T} Sa(\pi f \tau) \sum_k \delta(f - k/T)$$

$$= \sum_k \left[\left(\tau/T\right) Sa(\pi f \tau)\right] \delta(f - k/T)$$

$$= \sum_k \underbrace{\left[\left(\tau/T\right) Sa\left(\pi \frac{k\tau}{T}\right)\right]}_{C_k \equiv \text{ FS coefficient}} \delta(f - k/T)$$

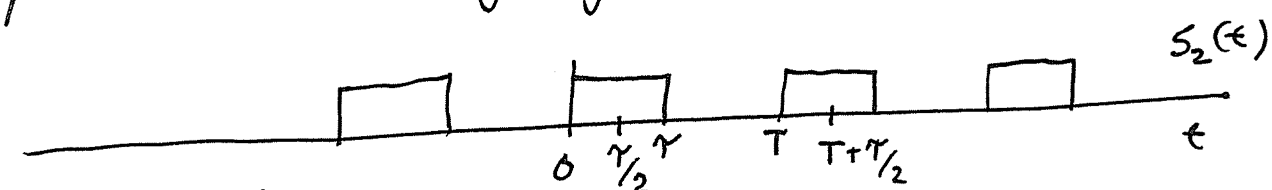
Then $s(t) = \mathcal{F}_f^{-1}\{S(f)\}$

$$= \int_{-\infty}^{\infty} \sum_k \left[\left(\tau/T\right) Sa\left(\pi \frac{k\tau}{T}\right)\right] \delta(f - k/T)\, e^{j2\pi f t}\, df$$

$$= \sum_k \left[\left(\tau/T\right) Sa\left(\pi \frac{k\tau}{T}\right)\right] \underbrace{\int_{-\infty}^{\infty} \delta(f - k/T)\, e^{j2\pi f t}\, df}_{\text{samples @ } f = k/T}$$

$$= \sum_k \left[\left(\tau/T\right) Sa\left(\pi \frac{k\tau}{T}\right)\right] e^{j2\pi \frac{kt}{T}} = \text{FS of } s(t)$$

Examples: 50% duty cycle and align the $s(t)$ to be



50% duty yields $\tau/T = 0.5$ so $\tau = T/2$

what about the time shift $s_2(t) = s(t - T/4)$

so $s_2(t) = \text{rect}\left(\frac{t - T/4}{T}\right) * \sum_n \delta(t - T/4 - nT)$

$$S_2(f) = e^{-j 2\pi \frac{T}{4} f}\, S(f)$$

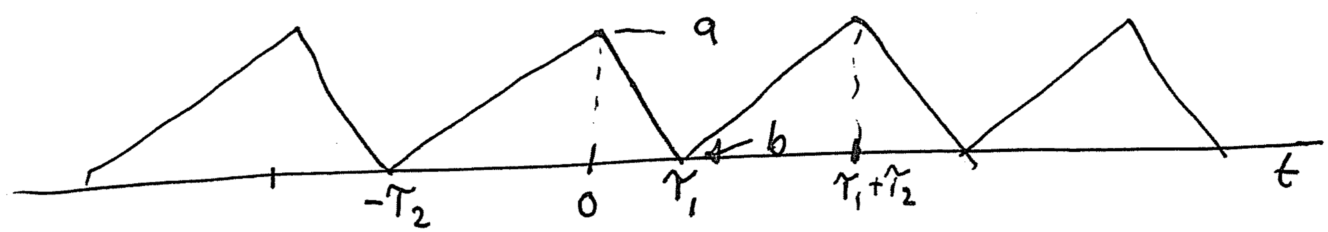
$$= e^{-j 2\pi \frac{T}{4} f} \left(\frac{\tau}{T}\right) Sa(\pi f \tau) \sum_k \delta(k - k/T)$$

where $\tau = T/2$

Where is the FS prefered over the FT?

When the periodic waveform is complicated and/or the FT is not known.

An example of a difficult function is the sawtooth waveform



$$C_n = \frac{1}{\tau_1+\tau_2}\left[\underbrace{\int_{-\tau_2}^{0}(a_2 t+b_2)\,e^{-j2\pi f_0 n t}\,dt + \int_{0}^{\tau_1}(a_1 t+b_1)\,e^{-j2\pi f_0 n t}\,dt}_{\text{piece wise integration}}\right]$$

We can fine a_1, b_1 from $t=0$ $a=b_1$, $a=b_2$

$t=\tau_1$ $0=a_1\tau_1+b_1$ so $a_1 = -a/\tau_1$

$t=-\tau_2$ $0=a_2(-\tau_2)+b_2$ $a_2 = a/\tau_2$

Break integrals up into 2 integrals each

$$\int_{-\tau_2}^{0} a_2 t\, e^{-j2\pi f_0 n t}\,dt + \int_{-\tau_2}^{0} b_2\, e^{-j2\pi f_0 n t}\,dt$$

Use integral table to obtain the anti-derivative and evaluate at the limits.