Lecture 5 proof of convolution.

Let's start with inverse $\mathcal{F}^{-1}\{\}$

i.e. $\quad ? = \int_{-\infty}^{\infty} W_1(f)\, W_2(f)\, e^{j2\pi f t}\, df$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_1(\lambda)\, e^{-j2\pi f\lambda}\, d\lambda \int_{-\infty}^{\infty} w_2(\beta)\, e^{-j2\pi f\beta}\, d\beta\; e^{j2\pi f t}\, df$$

$$= \iint_{-\infty}^{\infty} w_1(\lambda)\, w_2(\beta) \left[ \underbrace{\underbrace{\int_{-\infty}^{\infty} e^{-j2\pi f\lambda}\, e^{j2\pi f t}\, e^{-j2\pi f\beta}\, df}_{\int_{-\infty}^{\infty} e^{-j2\pi f(\lambda+\beta)}\, e^{j2\pi f t}\, df}}_{\delta(t-(\lambda+\beta))} \right] d\lambda\, d\beta$$

$$= \iint_{-\infty}^{\infty} w_1(\lambda) W_2(\beta)\, \underbrace{\delta(t-(\lambda+\beta))}_{\delta((t-\beta)-\lambda)}\, d\lambda\, d\beta \quad \text{samples when } \lambda = t-\beta$$

$$\boxed{w_1(t) * w_2(t) = \int_{-\infty}^{\infty} w_1(t-\beta)\, w_2(\beta)\, d\beta}$$

Multiplication

$$w_1(t)\,w_2(t) \longleftrightarrow W_1(f) * W_2(f)$$

We use the convolution property to make ~~an~~ numerically efficient approximations using the Fast Fourier Transform (FFT)

i.e. $$y[n] = x_1[n] * x_2[n] = \mathrm{FFT}^{-1}\left(\mathrm{FFT}(x_1[n]) .* \mathrm{FFT}(x_2[n])\right)$$

which approximates

$$y(t) = x_1(t) * x_2(t) = \mathcal{F}^{-1}\left\{ X_1(f)\, X_2(f) \right\}$$