Throughput Analysis in Automotive Paint Shops: A Case Study

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***Abstract*—In this paper, an overlapping decomposition method is used to estimate the throughput of a production system with multiple rework loops. The idea of the method is to decompose the system into a couple of serial lines and modify the parameters of overlapping machines to accommodate the effects of other lines. Using this method, the throughput of an automotive paint shop is analyzed and continuous improvement procedures are described.**

***Note to Practitioners*—Painting is an important element of vehicle production. A paint shop has been a system bottleneck in many automotive assembly plants due to its complexity. Fast and accurate analysis of its system throughput is important for design and continuous improvements. This paper introduces an iterative method to analyze the performance of paint shop type production systems, i.e., systems with multiple rework loops. The method has obtained good results in both theoretical study and applications on the factory floor. In addition, a case study at an automotive paint shop is introduced and continuous improvements process to identify and eliminate system bottlenecks is described. The presented method can also be applied to other production systems with similar structures.**

***Index Terms*—Overlapping decomposition, paint shop, rework loop, throughput analysis.**

I. Introduction

Painting is an important element of vehicle production. A paint shop is a system bottleneck in many automotive assembly plants due to complexity inherent in the process, production control policies and rigorous quality requirements [1]. In automotive paint shops, rework loops are often required when a job needs multiple passes or is defective. Jobs can enter the painting booths multiple times, either for repaint or for "tutone" operation (i.e., to have different colors painted). The use of rework loops in paint shops can significantly increase the system throughput and reduce scrap, cost, etc. To design, operate, and improve the performance of paint shops, accurate throughput analysis, which is a critical enabler for continuous improvement is necessary and important.

Throughput analysis of production systems has attracted intensive attention (see reviews [2], [3] and books [4]–[6]). For two-machine systems, there exist exact analytical solutions. For longer lines and assembly systems, different aggregation and decomposition techniques have been used to approximate system performance. However, in spite of all these efforts, the results on production systems with rework loops, i.e., paint-shop-type systems, is quite limited.

Although some analytical methods have been developed to analyze closed-loop queueing networks (see reviews [7] and [8]), most of them do not address the unreliable nature of a production system and do not study the rework type system directly. There are a limited number of studies on throughput analysis in closed-loop production systems or paint shops. Closed-loop serial production lines with a fixed number of carriers, where parts are loaded on recirculating pallets at the first machine and unloaded at the last after the part has undergone all the required operations, have been studied in [9]–[12]. Specifically, paper [9] analyzes an asymptotically reliable two-machine two-buffer closed-loop line and describes a case study in a paint shop. Paper [10] presents a decomposition approach for longer homogeneous production lines and investigates the optimal number of carriers which maximizes system throughput. Paper [11] introduces thresholds for blocking and starving probabilities to take into account the correlation between number of parts in the buffer and uses loop transformations to decompose the system into two-machine building blocks to estimate the performance of systems with both small and large loops. In addition, by using Taylor series expansions, an approximation method is described in [12] for highly reliable closed-loop systems.

However, for production systems with unequal machine speeds and multiple rework loops (which are common in paint shops), to our best knowledge, there are no analytical methods available in the literature to analyze their performance. The main contribution of this paper is an iterative approach to estimate the throughput of such systems and illustrate its applicability through a continuous improvement project at an automotive paint shop.

To this end, the remainder of the paper is structured as follows: Sections II formulates the problem. The approach of throughput analysis is presented in Section III. Using this approach, Section IV introduces a case study at an automotive paint shop. The conclusions are formulated in Section V. All proofs are given in the Appendix.

II. Problem Formulation

A typical structure of an automotive paint shop is shown in Fig. 1, where the circles represent machines and the rectangles are buffers. The system consists of a main line and three rework loops, including repair, tutone and polishing (denoted as subscripts r, t, and p, respectively). A description of notations of machines and buffers is introduced as follows:

$$
\begin{aligned}
\text{Main line:}\quad & m_1,\dots,m_M,\quad B_1,\dots,B_{M-1}\\
\text{Repair loop:}\quad & m_{r1},\dots,m_{rR},\quad B_{r1},\dots,B_{rR+1}\\
\text{Tutone loop:}\quad & m_{t1},\dots,m_{tT},\quad B_{t1},\dots,B_{tT+1}\\
\text{Polishing loop:}\quad & m_{p1},\dots,m_{pP},\quad B_{p1},\dots,B_{pP+1}\\
\text{Merge machine:}\quad & m_{j_r}(\text{repair/tutone merge})\\
& m_{j_p}(\text{polishing merge})\\
\text{Split machine:}\quad & m_{k_r}(\text{rework split}),\quad m_{k_t}(\text{tutone split})\\
& m_{rk_p}(\text{polishing split}).
\end{aligned}
$$

A defective part is sent to the repair loop at machine m_{k_r}. Then, at machine m_{rk_p}, the parts needing minor repairs are corrected in the polishing loop and merged with the main line at machine m_{j_p}, whereas the severe defective parts are stayed in the repair loop and back to the main line at machine m_{j_r} for repaint. In addition, some parts may need different colors, they are routed to the tutone loop at machine m_{k_t}. The tutone parts are joined with the main line at machine m_{j_r} for another paint.

The following model is considered throughout this paper.

i) Each machine m_i has two states: up and down. When up, the machine is capable of producing with the rate S_i parts per unit of time; when the machine is down, no production takes place.
ii) The up and downtimes of each machine m_i, are exponentially distributed with parameters p_i and r_i, respectively. In other words, p_i and r_i are the failure and repair rates, respectively.

Remark 1: Assumption ii) implies that $T_{\rm up}(i)$ and $T_{\rm down}(i)$, the average up and downtimes of m_i, equal to $(1/p_i)$ and $(1/r_i)$, respectively. The exponential distribution is used to simplify the analysis. This

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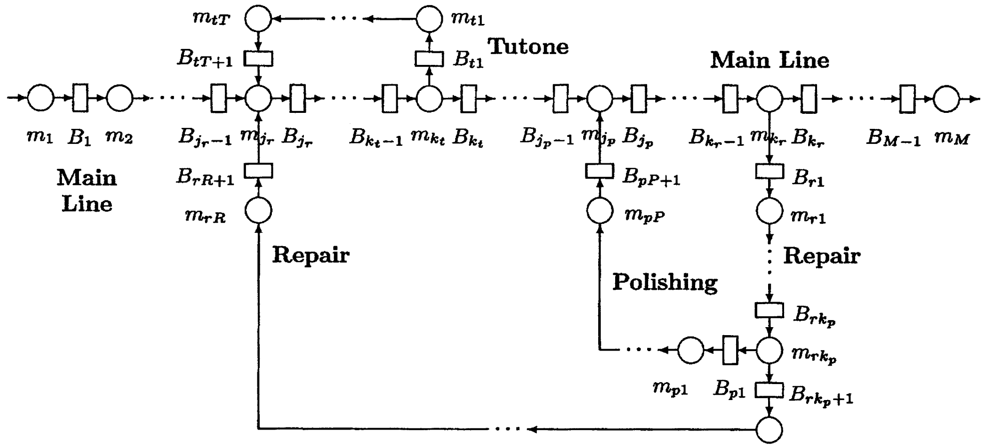

Fig. 1. Typical structure of an automotive paint shop.

assumption will be relaxed in future work. A validation of the exponential assumption in production system research is investigated in [13].

iii) Each buffer B_i has a capacity $N_i, 0 \leq N_i < \infty$.

iv) At machine m_{k_r}, a part is defective with probability α_r, and is sent to repair loop. Similarly, a part has probabilities α_t or α_p routing to tutone or polishing loops, respectively. Probabilities α_r, α_t, and α_p are referred to as the rework, tutone, and polishing rates, respectively. All $\alpha_i \in [0, 1)$.

v) Machine m_{j_r} takes part from B_{rR+1} first if it is not empty, B_{tT+1} second, and finally B_{j_r-1}. Similarly, m_{j_p} takes part from polishing loop first.

vi) Machine m_i is starved at time t if the upstream buffer B_{i-1} is empty at time t. Machine m_1 is never starved.

vii) Machine m_i is blocked at time t if downstream buffer B_i is full at time t. Machine m_M is never blocked. In particular, m_{k_r} is blocked by main line if it produces a good part and B_{k_r} is full. m_{k_r} is blocked by repair loop if it produces a defective part and B_{r1} is full. Similarly, m_{k_t} and m_{rk_p} can be blocked by main/tutone lines and repair/polishing lines, respectively.

The problem addressed in this paper is as follows.*Given production system i)–vii), develop a method for evaluating the throughput as a function of the system parameters.*

A solution to the problem is given next.

III. Approach of Throughput Analysis

A. Idea of Overlapping Decomposition

Due to the complexity in the system with rework loops, direct analysis is not possible. Therefore, we introduce an iterative approach, referred to as *overlapping decomposition*. The idea of the approach is to decompose the system into serial lines, where the first and last machines in one serial line are overlapped with another serial line, and are modified to accommodate the effects of other lines (see Fig. 2). More specifically, consider a serial production line (for instance, lines 1 or 5). A recursive procedure to evaluate its throughput is developed in [14] (see Subsection III.B2). However, it assumes the first and last machines are not starved and blocked, respectively. If we know the probabilities that the first and last machines are starved and blocked, respectively, we can introduce fictitious machines to accommodate these probabilities and use this procedure to calculate the throughput of this line and the throughput of the whole system. Since these probabilities are unknown, we introduce iterations, as described below.

As the first step, consider line 6 (repair) in Fig. 2. Assume that we know $\Pr\{m_{k_r}$ is starved, $\Pr\{m_{rk_p}$ is blocked by polishing$\}$, and $\Pr\{m_{rk_p}$ is blocked by repair$\}$, introduce fictitious machines m'_{k_r} and m'_{rk_p} with parameters defined by

$$\text{Line 6:} \quad r'_{k_r} = r_{k_r}\alpha_r\left(1 - \Pr\{m_{k_r}\text{is starved}\}\right)$$
$$r'_{rk_p} = r_{rk_p}\left[1 - \alpha_p \Pr\left\{m_{rk_p}\text{is blocked by polishing}\right\} - \left(1-\alpha_p\right)\Pr\left\{m_{rk_p}\text{is blocked by repair}\right\}\right]$$

where r'_{k_r} is introduced by considering the starvation time and the time to produce a good part as machine downtime (from the point of view of the downstream machine). Therefore, the average machine downtime $T_{\text{down}}(k'_r)$ can be equivalent to $(T_{\text{down}}(k_r))/(\alpha_r \Pr\{m_{k_r}\text{is not starved}\})$. (Analogously, blockage time can also be considered as downtime from the point of view of upstream machines). A similar equivalence is used for r'_{rk_p} and subsequent modifications. Select p'_{k_r} and p'_{rk_p} by following the conservation of flow [6] such that

$$\frac{r'_{k_r}}{r'_{k_r} + p'_{k_r}} = \frac{r_{k_r}\alpha_r}{r_{k_r} + p_{k_r}} \Pr\{m_{k_r}\text{is not starved}\}$$

which implies that $p'_{k_r} = p_{k_r} + r_{k_r} - r'_{k_r}$. Similar adjustments are used to select p'_{rk_p} and other p'_i's in all subsequent discussions when modifying the first and last machines.

Use the serial line evaluation procedure, we can calculate $\Pr\{m_{rk_p}$ is starved$\}$ and $\Pr\{m_{k_r}$ is blocked by repair$\}$. The detailed calculation procedure is defined in Subsection III.B.2. Now, consider line 7 (repair). Assume that $\Pr\{m_{j_r}$ is blocked$\}$ is known and modify machines m_{rk_p} and m_{j_r} to fictitious machines m''_{rk_p} and m'_{j_r}, respectively, with the parameters defined by

$$\text{Line 7:} \quad r''_{rk_p} = r_{rk_p}(1-\alpha_p)\left(1 - \Pr\left\{m_{rk_p}\text{ is starved}\right\}\right)$$
$$r'_{j_r} = r_{j_r}\left(1 - \Pr\{m_{j_r}\text{ is blocked}\}\right).$$

Now, calculate $\Pr\{m_{j_r}$ is starved by repair$\}$ and $\Pr\{m_{rk_p}$ is blocked by repair$\}$. Analogously, consider lines 9 (tutone) and 8 (polishing), assume $\Pr\{m_{k_t}$ is starved$\}$ and $\Pr\{m_{j_p}$ is blocked$\}$ are given, introduce fictitious machines m'_{k_t}, m''_{j_r}, and m'''_{rk_p}, m'_{j_p}, respectively, and let

$$\text{Line 9:} \quad r'_{k_t} = r_{k_t}\alpha_t\left(1 - \Pr\{m_{k_t}\text{ is starved}\}\right)$$
$$r''_{j_r} = r_{j_r}\left(1 - \Pr\{m_{j_r}\text{ is blocked}\}\right)$$
$$\Pr\{m_{j_r}\text{ is starved by repair}\}$$
$$\text{Line 8:} \quad r'''_{rk_p} = r_{rk_p}\alpha_p\left(1 - \Pr\left\{m_{rk_p}\text{ is starved}\right\}\right)$$
$$r'_{j_p} = r_{j_p}\left(1 - \Pr\left\{m_{j_p}\text{ is blocked}\right\}\right).$$

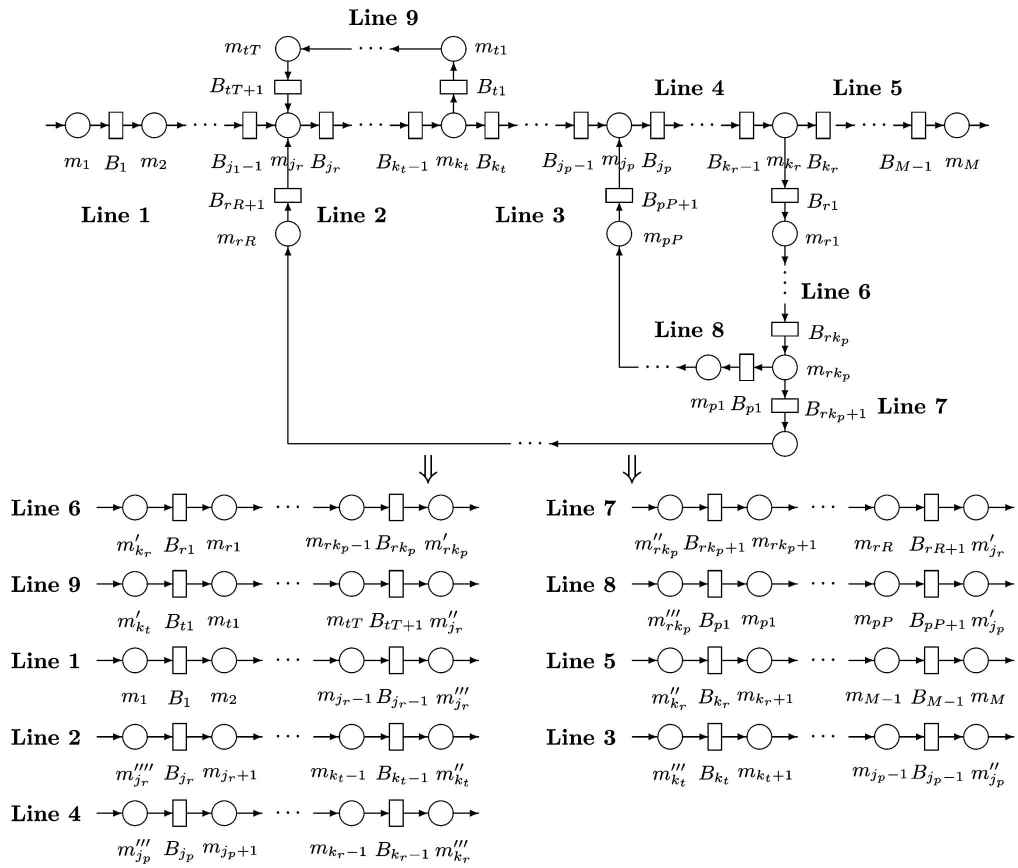

Fig. 2. Overlapping decomposition of system i)–vii) into lines 1–9.

Then, probabilities $\Pr\{m_{j_r}$ is starved by tutone$\}$ and $\Pr\{m_{k_t}$ is blocked by tutone$\}$, $\Pr\{m_{j_p}$ is starved by polishing$\}$, and $\Pr\{m_{rk_p}$ is blocked by polishing$\}$ can be calculated. This completes the analysis of rework loops. Next, we study the main line. Consider lines 5 and 1, adjust machines m_{k_r} and m_{j_r} to m''_{k_r} and m'''_{j_r}, respectively, and define

$$\text{Line 5:}\quad r''_{k_r} = r_{k_r}(1-\alpha_r)\left(1-\Pr\{m_{k_r}\text{ is starved}\}\right)$$
$$\text{Line 1:}\quad r'''_{j_r} = r_{j_r}\left(1-\Pr\{m_{j_r}\text{ is blocked}\}\right)\Pr\{m_{j_r}\text{ is starved by repair and tutone}\}.$$

Thus, probabilities $\Pr\{m_{k_r}$ is blocked by main line$\}$ and $\Pr\{m_{j_r}$ is starved by main$\}$ line can be obtained. Now consider line 2, at first assume $\Pr\{m_{k_t}$ is blocked by main line$\}$ is known, modify machines m_{j_r} and m_{k_t} to m''''_{j_r} and m''_{k_t}, i.e.,

$$\text{Line 2:}\quad r''''_{j_r} = r_{j_r}\left[1-\Pr\{m_{j_r}\text{ is starved by main line, repair and tutone}\}\right]$$
$$r''_{k_t} = r_{k_t}\left(1-\alpha_t\Pr\{m_{k_t}\text{ is blocked by tutone}\} - (1-\alpha_t)\Pr\{m_{k_t}\text{ is blocked by main line}\}\right).$$

Calculate probabilities $\Pr\{m_{j_r}$ is blocked$\}$ and $\Pr\{m_{k_t}$ is starved$\}$. Finally, for lines 3 and 4, using the probabilities calculated above, modify machines m_{k_t} and m_{j_p} (line 3), m_{j_p} and m_{k_r} (line 4), to m'''_{k_t} and m''_{j_p}, m'''_{j_p} and m'''_{k_r}, respectively. We obtain

$$\text{Line 3:}\quad r'''_{k_t} = r_{k_t}\left(1-\Pr\{m_{k_t}\text{ is starved}\}\right)$$
$$r''_{j_p} = r_{j_p}\left(1-\Pr\{m_{j_p}\text{ is blocked}\}\right)\Pr\{m_{j_p}\text{ is starved by polishing}\}$$

$$\text{Line 4:}\quad r'''_{j_p} = r_{j_p}\left(1-\Pr\{m_{j_p}\text{ is starved by main line and polishing}\}\right)$$
$$r'''_{k_r} = r_{k_r}\left(1-\alpha_r\Pr\{m_{k_r}\text{ is blocked by repair}\} - (1-\alpha_r)\Pr\{m_{k_r}\text{ is blocked by main line}\}\right).$$

Probabilities $\Pr\{m_{k_t}$ is blocked by main line$\}$ and $\Pr\{m_{j_p}$ is starved by main line$\}$, $\Pr\{m_{j_p}$is blocked$\}$, and $\Pr\{m_{k_r}$ is starved$\}$ can be calculated, respectively. Use now these probabilities for the second iteration in analysis of line 6 and continue this process, alternating among all 9 lines. A graphic illustration of this procedure is shown in Fig. 3. As shown below, the iterations are convergent and result in the estimate of system throughput. A formal expression of this procedure is introduced below.

B. Recursive Procedures

1) Iterations: For simplification, introduce the following notations:

$$s_i = \Pr\{m_i\text{ is starved}\},\qquad i = k_r, k_t, rk_p$$
$$s_{ix} = \Pr\{m_i\text{ is starved by main line }(x=m)\text{, repair }(x=r)\text{, tutone }(x=t)\text{ and polishing }(x=p)\text{, respectively}\},\qquad i = j_r, j_p,$$
$$b_i = \Pr\{m_i\text{ is blocked}\}),\qquad i = j_r, j_p,$$
$$b_{ix} = \Pr\{m_i\text{ is blocked by main line }(x=m)\text{, repair }(x=r)\text{, tutone }(x=t)\text{ and polishing }(x=p)\text{, respectively}\},\qquad i = k_r, k_t, rk_p.$$

Let Operators Φ_1 and Φ_2 denote the procedures to calculate the probabilities that the first machine is blocked and the last machine is starved

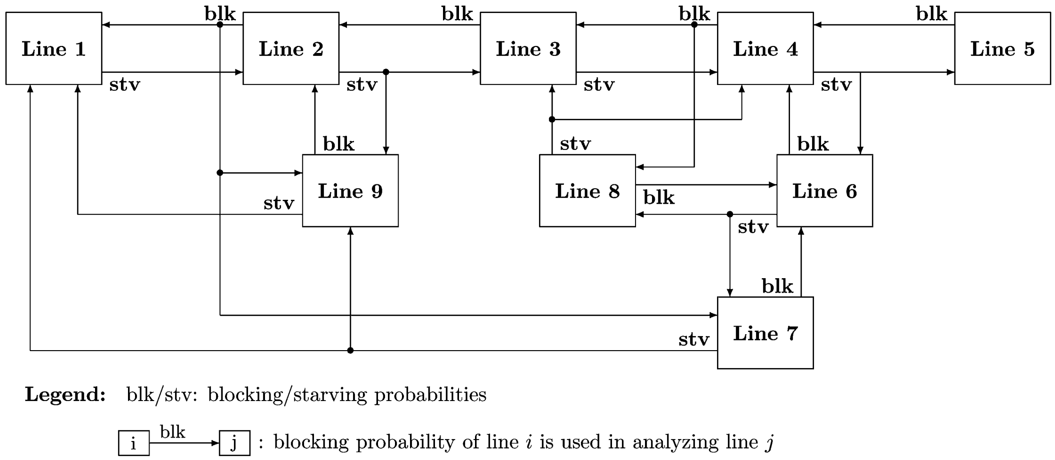

Fig. 3. Iterations among lines 1–9.

in a serial line, respectively. A formal definition of Φ_1 and Φ_2 is introduced in the next section. Formally, the recursive procedure is as follows.

Procedure 1:

Line 6: $(m_{k_r}, m_{r1}, \ldots, m_{rk_p})$

$$\begin{aligned}
r'_{k_r}(n+1) &= r_{k_r}\alpha_r\left[1 - s_{k_r}(n)\right],\\
p'_{k_r}(n+1) &= p_{k_r} + r_{k_r} - r'_{k_r}(n+1)\\
r'_{rk_p}(n+1) &= r_{rk_p}\left[1 - \alpha_p b_{rk_p p}(n) - (1-\alpha_p) b_{rk_p r}(n)\right]\\
p'_{rk_p}(n+1) &= p_{rk_p} + r_{rk_p} - r'_{rk_p}(n+1)\\
b_{k_r r}(n+1) &= \Phi_1\big(p'_{k_r}(n+1) r'_{k_r}(n+1), S_{k_r}, \ldots,\\
&\qquad p'_{rk_p}(n+1), r'_{rk_p}(n+1), S_{rk_p}, N_{r1}, \ldots, N_{rk_p}\big)\\
s_{rk_p}(n+1) &= \Phi_2\big(p'_{k_r}(n+1), r'_{k_r}(n+1), S_{k_r}, \ldots,\\
&\qquad p'_{rk_p}(n+1), r'_{rk_p}(n+1), S_{rk_p}, N_{r1}, \ldots, N_{rk_p}\big).
\end{aligned} \tag{1}$$

Line 7: $(m_{rk_p}, m_{rk_p+1}, \ldots, m_{rR}, m_{j_r})$

$$\begin{aligned}
r''_{rk_p}(n+1) &= r_{rk_p}(1-\alpha_p)\left[1 - s_{rk_p}(n+1)\right]\\
p''_{rk_p}(n+1) &= p_{rk_p} + r_{rk_p} - r''_{rk_p}(n+1)\\
r'_{j_r}(n+1) &= r_{j_r}\left[1 - b_{j_r}(n)\right]\\
p'_{j_r}(n+1) &= p_{j_r} + r_{j_r} - r'_{j_r}(n+1)\\
b_{rk_p r}(n+1) &= \Phi_1\big(p''_{rk_p}(n+1), r''_{rk_p}(n+1), S_{rk_p}, \ldots,\\
&\qquad p'_{j_r}(n+1), r'_{j_r}(n+1), S_{j_r}, N_{rk_p+1}, \ldots, N_{rR+1}\big)\\
s_{j_r r}(n+1) &= \Phi_2\big(p''_{rk_p}(n+1), r''_{rk_p}(n+1), S_{rk_p}, \ldots,\\
&\qquad p'_{j_r}(n+1), r'_{j_r}(n+1), S_{j_r}, N_{rk_p+1}, \ldots, N_{rR+1}\big).
\end{aligned} \tag{2}$$

Line 9: $(m_{k_t}, m_{t1}, \ldots, m_{tT}, m_{j_r})$

$$\begin{aligned}
r'_{k_t}(n+1) &= r_{k_t}\left[1 - s_{k_t}(n)\right]\alpha_t\\
p'_{k_t}(n+1) &= p_{k_t} + r_{k_t} - r'_{k_t}(n+1)\\
r''_{j_r}(n+1) &= r_{j_r}\left[1 - b_{j_r}(n)\right] s_{j_r r}(n+1)\\
p''_{j_r}(n+1) &= p_{j_r} + r_{j_r} - r''_{j_r}(n+1)\\
b_{k_t t}(n+1) &= \Phi_1\big(p'_{k_t}(n+1), r'_{k_t}(n+1), S_{k_t}, \ldots,\\
&\qquad p''_{j_r}(n+1), r''_{j_r}(n+1), S_{j_r}, N_{t1}, \ldots, N_{tT+1}\big)\\
s_{j_r t}(n+1) &= \Phi_2\big(p'_{k_t}(n+1), r'_{k_t}(n+1), S_{k_t}, \ldots,\\
&\qquad p''_{j_r}(n+1), r''_{j_r}(n+1), S_{j_r}, N_{t1}, \ldots, N_{tT+1}\big).
\end{aligned} \tag{3}$$

Line 8: $(m_{rk_p}, m_{p1}, \ldots, m_{pP}, m_{j_p})$

$$\begin{aligned}
r'''_{rk_p}(n+1) &= r_{rk_p}\left[1 - s_{rk_p}(n+1)\right]\alpha_p\\
p'''_{rk_p}(n+1) &= p_{rk_p} + r_{rk_p} - r'''_{rk_p}(n+1)\\
r'_{j_p}(n+1) &= r_{j_p}\left[1 - b_{j_p}(n)\right]\\
p'_{j_p}(n+1) &= p_{rk_p} + r_{j_p} - r'_{j_p}(n+1)\\
s_{j_p p}(n+1) &= \Phi_2\big(p'''_{rk_p}(n+1), r'''_{rk_p}(n+1), S_{rk_p}, \ldots,\\
&\qquad p'_{j_p}(n+1), r'_{j_p}(n+1), S_{j_p}, N_{p1}, \ldots, N_{pP+1}\big).
\end{aligned} \tag{4}$$

Line 5: $(m_{k_r}, m_{k_r+1}, \ldots, m_M)$

$$\begin{aligned}
r''_{k_r}(n+1) &= r_{k_r}\left[1 - s_{k_r}(n)\right](1-\alpha_r)\\
p''_{k_r}(n+1) &= p_{k_r} + r_{k_r} - r''_{k_r}(n+1)\\
b_{k_r m}(n+1) &= \Phi_1\big(p''_{k_r}(n+1), r''_{k_r}(n+1), S_{k_r}, \ldots,\\
&\qquad p_M, r_M, S_M, N_{k_r}, \ldots, N_{M-1}\big).
\end{aligned} \tag{5}$$

Line 1: $(m_1, \ldots, m_{j_r})$

$$\begin{aligned}
r'''_{j_r}(n+1) &= r_{j_r}\left[1 - b_{j_r}(n)\right] s_{j_r r}(n+1) s_{j_r t}(n+1)\\
p'''_{j_r}(n+1) &= p_{j_r} + r_{j_r} - r'''_{j_r}(n+1)\\
s_{j_r m}(n+1) &= \Phi_1\big(p_1, r_1, S_1, \ldots, p'''_{j_r}(n+1)\\
&\qquad r'''_{j_r}(n+1), S_{j_r}, N_1, \ldots, N_{j_r-1}\big).
\end{aligned} \tag{6}$$

Line 2: $(m_{j_r}, m_{j_r+1}, \ldots, m_{k_t})$

$$\begin{aligned}
r''''_{j_r}(n+1) &= r_{j_r}\left[1 - s_{j_r m}(n+1) s_{j_r t}(n+1) s_{j_r r}(n+1)\right]\\
p''''_{j_r}(n+1) &= p_{j_r} + r_{j_r} - r''''_{j_r}(n+1)\\
r''_{k_t}(n+1) &= r_{k_t}\left[1 - \alpha_t b_{k_t t}(n+1) - (1-\alpha_t) b_{k_t m}(n)\right]\\
p''_{k_t}(n+1) &= p_{k_t} + r_{k_t} - r'''_{k_t}(n+1)\\
s_{k_t}(n+1) &= \Phi_2\big(p''''_{j_r}(n+1), r''''_{j_r}(n+1), S_{j_r}, \ldots,\\
&\qquad p''_{k_t}(n+1), r''_{k_t}(n+1), S_{k_t}, N_{j_r}, \ldots, N_{k_t-1}\big).
\end{aligned} \tag{7}$$

Line 3: $(m_{k_t}, m_{k_t+1}, \ldots, m_{j_p})$

$$\begin{aligned}
r'''_{k_t}(n+1) &= r_{k_t}\left[1 - s_{k_t}(n+1)\right](1-\alpha_t)\\
p'''_{k_t}(n+1) &= p_{k_t} + r_{k_t} - r'''_{k_t}(n+1)\\
r''_{j_p}(n+1) &= r_{j_p}\left[1 - b_{j_p}(n)\right] s_{j_p p}(n+1)\\
p''_{j_p}(n+1) &= p_{rk_p} + r_{j_p} - r''_{j_p}(n+1)\\
s_{j_p m}(n+1) &= \Phi_2\big(p'''_{k_t}(n+1), r'''_{k_t}(n+1), S_{k_t}, \ldots,\\
&\qquad p''_{j_p}(n+1), r''_{j_p}(n+1), S_{j_p}, N_{k_t}, \ldots, N_{j_p-1}\big)
\end{aligned} \tag{8}$$

$$
\begin{aligned}
&\text{Line 4:} \qquad (m_{j_p}, m_{j_p+1}, \ldots, m_{k_r})\\
&r'''_{j_p}(n+1) = r_{j_p}\left[1 - s_{j_p m}(n+1) s_{j_p p}(n+1)\right]\\
&p'''_{j_p}(n+1) = p_{j_p} + r_{j_p} - r'''_{j_p}(n+1)\\
&r'''_{k_r}(n+1) = r_{k_r}\left[1 - (1-\alpha_r) b_{k_r m}(n+1) - \alpha_r b_{k_r r}(n+1)\right]\\
&p'''_{k_r}(n+1) = p_{k_r} + r_{k_r} - r'''_{k_r}(n+1)\\
&b_{j_p}(n+1) = \Phi_1(p'''_{j_p}(n+1), r'''_{j_p}(n+1), S_{j_p}, \ldots, p'''_{k_r}(n+1), r'''_{k_r}(n+1), S_{k_r}, N_{j_p}, \ldots, N_{k_r-1})\\
&s_{k_r}(n+1) = \Phi_2\left(p'''_{j_p}(n+1), r'''_{j_p}(n+1), S_{j_p}, \ldots, p'''_{k_r}(n+1), r'''_{k_r}(n+1), S_{k_r}, N_{j_p}, \ldots, N_{k_r-1}\right).
\end{aligned} \tag{9}
$$

The initial conditions are

$$
\begin{aligned}
&s_i(0) = 0, \qquad i = k_r, k_t,\\
&b_i(0) = 1, \qquad i = j_r, j_p, k_t m, r k_p p, r k_p r.
\end{aligned}
$$

2) Operators Φ_1 and Φ_2: Operators Φ_1 and Φ_2 are defined through the aggregation procedure for performance analysis of serial lines developed in [14]. Consider a serial production line consisting of M machines with parameters $p_1, r_1, S_1, \ldots, p_M, r_M, S_M$, and $M-1$ in-process buffers with capacities $N_1, \ldots, N_{M-1}$. According to [14], its performance can be analyzed using the recursive procedure introduced below (see [14] for details).

Basically, the procedure consists of two aggregations: the forward aggregation and backward aggregation. In the forward aggregation, the first two machines are aggregated into a single machine, m_2^f. Next, m_2^f is aggregated with m_3 to result in m_3^f, and so on, until all M machines are aggregated into a single one m_M^f. Then, in the backward aggregation, the last machine m_M is aggregated with m_{M-1}^f to generate m_{M-1}^b and so on until all machines are again aggregated into a single machine m_1^b. Then, the procedure is repeated again. In addition, parameters ρ and ν are introduced to characterize the mean and variance of the throughput of a one-machine system, which can be used to determine the aggregated machine parameters, and b_i^b and s_i^f are blockage and starvation parameters, respectively. Formally, this process can be represented as follows.

Procedure 2:

Backward aggregation $\quad i = M-1, \ldots, 1$

$$
\begin{aligned}
&b_i^b(l+1) = 1 - \mathrm{TP}_2\left(p_i^f(l), r_i^f(l), S_i^f(l), p_{i+1}^b(l+1), r_{i+1}^b(l+1), S_{i+1}^b(l+1), N_i\right) \Big/ \rho_i^f(l)\\
&\rho_i^b(l+1) = \rho_i\left[1 - b_i^b(l+1)\right]\\
&\nu_i^b(l+1) = \nu_i\left[1 - b_i^b(l+1)\right] + v_{i+1}^b(l+1) b_i^b(l+1)\\
&S_i^b(l+1) = \begin{cases} S_i, & \text{if } S_{i+1}^b(l+1) \geq S_i\\ S_i\left[1 - b_i^b(l+1)e_i\right] + S_{i+1}^b(l+1)\\ \quad b_i^b(l+1)e_i, & \text{if } S_{i+1}^b(l+1) < S_i \end{cases}\\
&r_i^b(l+1) = \frac{2\left[\rho_i^b(l+1)\right]^2\left[S_i^b(l+1) - \rho_i^b(l+1)\right]}{S_i^b(l+1)\nu_i^b(l+1)}\\
&p_i^b(l+1) = \frac{2\rho_i^b(l+1)\left[S_i^b(l+1) - \rho_i^b(l+1)\right]^2}{S_i^b(l+1)\nu_i^b(l+1)}.
\end{aligned} \tag{10}
$$

Forward aggregation $\quad i = 1, \ldots, M-1$

$$
\begin{aligned}
&s_{i+1}^f(l+1) = 1 - \mathrm{TP}_2\left(p_i^f(l+1), r_i^f(l+1), S_i^f(l+1), p_{i+1}^b(l+1), r_{i+1}^b(l+1), S_{i+1}^b(l+1), N_i\right) \Big/ \rho_{i+1}^b(l+1)\\
&\rho_{i+1}^f(l+1) = \rho_{i+1}\left[1 - s_{i+1}^f(l+1)\right]\\
&\nu_{i+1}^f(l+1) = \nu_{i+1}\left[1 - s_{i+1}^f(l+1)\right] + v_i^f(l+1) s_{i+1}^f(l+1)\\
&S_{i+1}^f(l+1) = \begin{cases} S_{i+1}, & \text{if } S_i^f(l+1) \geq S_{i+1}\\ S_{i+1}\left[1 - s_{i+1}^f(l+1)e_{i+1}\right] + S_i^f(l+1)\\ \quad s_{i+1}^f(l+1)e_{i+1}, & \text{if } S_i^f(l+1) < S_{i+1} \end{cases}\\
&r_{i+1}^f(l+1) = \frac{2\left[\rho_{i+1}^f(l+1)\right]^2\left[S_{i+1}^f(l+1) - \rho_{i+1}^f(l+1)\right]}{S_{i+1}^f(l+1)\nu_{i+1}^f(l+1)}\\
&p_{i+1}^f(l+1) = \frac{2\rho_{i+1}^f(l+1)\left[S_{i+1}^f(l+1) - \rho_{i+1}^f(l+1)\right]^2}{S_{i+1}^f(l+1)\nu_{i+1}^f(l+1)}
\end{aligned} \tag{11}
$$

where

$$
\begin{aligned}
e_i &= \frac{r_i}{p_i + r_i}\\
\rho_i &= S_i e_i\\
\nu_i &= \frac{2S_i^2 r_i p_i}{(r_i + p_i)^3}, \qquad i = 1, \ldots, M
\end{aligned} \tag{12}
$$

with boundary conditions

$$
\begin{aligned}
p_1^f(l) &= p_1\\
r_1^f(l) &= r_1\\
S_1^f(l) &= S_1, \rho_1^f(l)\\
&= S_1 e_1\\
p_M^b(l) &= p_M\\
r_M^b(l) &= r_M\\
S_M^f(l) &= S_M\\
\rho_M^b(l) &= S_M e_M\\
\nu_1^f(l) &= \nu_1\\
\nu_M^b(l) &= \nu_M
\end{aligned}
$$

and initial conditions

$$
\begin{aligned}
p_i^f(0) &= p_i\\
r_i^f(0) &= r_i\\
S_i^f(0) &= S_i\\
\rho_i^f(0) &= S_i e_i\\
p_i^b(0) &= p_i\\
r_i^b(0) &= r_i\\
S_i^f(0) &= S_i\\
\rho_i^b(0) &= S_i e_i\\
\nu_i^f(0) &= \nu_i\\
\nu_i^b(0) &= \nu_i, \qquad i = 1, \ldots, M
\end{aligned}
$$

and $\mathrm{TP}_2(p_1, r_1, S_1, p_2, r_2, S_2, N)$ is defined by ([14] and [15])

- Case 1: $S_1 = S_2 = S$

$$
\mathrm{TP}_2 = \begin{cases} \dfrac{r_1 r_2 S[p_1(p_2 + r_2) - p_2(p_1 + r_1)e^{-\beta N}]}{(p_1 + r_1)(p_1 + r_2)(p_1 r_2 - p_2 r_1 e^{-\beta N})}, & \text{if } \dfrac{p_1}{r_1} \neq \dfrac{p_2}{r_2}\\[2ex] \dfrac{S r_2^2 (r_1 + r_2) + N r_1 r_2 (p_2 + r_2)^2}{(p_2 + r_2)[S(r_1 + r_2) + N r_1 (p_2 + r_2)]} S, & \text{if } \dfrac{p_1}{r_1} = \dfrac{p_2}{r_2} \end{cases} \tag{13}
$$

where

$$
\beta = \frac{(r_1 + r_2 + p_1 + p_2)(p_1 r_2 - p_2 r_1)}{(r_1 + r_2)(p_1 + p_2)S}.
$$

- Case 2: $S_1 < S_2$

$$\mathrm{TP}_2 = \frac{S_2 e_2 A e^{\gamma_1 N} + S_1 e_1 B e^{\gamma_2 N} + S_1 e_1 C e^{-\gamma_2 N}}{A e^{\gamma_1 N} + B e^{\gamma_2 N} + C e^{-\gamma_2 N}} \tag{14}$$

where

$$\begin{aligned} A &= r_1 Q^2 + r_1 Q[S_1(r_1 + r_2 + p_2) - S_2(r_1 + r_2 + p_1)] \\ B &= r_2 p_1 S_2[(S_1 - S_2)(r_1 - r_2) - (S_2 p_1 + S_1 p_2) - Q] \\ C &= \frac{e_2^2(S_2 - S_1 e_1)A + S_1 e_1(1 - S_2)B}{S_1 e_1(e_2 - 1)} \\ \gamma_1 &= \frac{1}{2S_1 S_2(r_1 + r_2)(S_1 - S_2)} \left[r_1 S_1^2(r_1 + r_2 + p_2)\right. \\ &\quad + r_2 S_2^2(r_1 + p_1 + r_2) - S_1 S_2[(r_1 + r_2)^2 \\ &\quad \left. + (r_1 + r_2)(p_1 + p_2) + (r_1 p_2 + r_2 p_1)]\right] \\ \gamma_2 &= \frac{(S_1 r_1 + S_2 r_2)Q}{2S_1 S_2(r_1 + r_2)(S_2 - S_1)} \\ Q &= \sqrt{[S_1(r_1 + r_2 + p_2) - S_2(r_1 + r_2 + p_1)]^2 + 4S_1 S_2 p_1 p_2}. \end{aligned} \tag{15}$$

- Case 3: $S_1 > S_2$ (by reversibility [16])

$$\mathrm{TP}_2 = \mathrm{TP}_2(p_2, r_2, S_2, p_1, r_1, S_1, N). \tag{16}$$

In terms of the steady state of the procedure, the throughput of the serial line $\widehat{\mathrm{TP}}$ and operators Φ_1 and Φ_2 are defined through ρ_M^f, b_1^b, and s_M^f, respectively

$$\begin{aligned} \widehat{\mathrm{TP}} &= \mathrm{TP}_M(p_1, r_1, S_1, \ldots, p_M, r_M, S_M, N_1, \ldots, N_{M-1}) \\ &= \lim_{l\to\infty} \rho_M^f(l) = \rho_M^f \\ b_1(n) &= \Phi_1(p_1, r_1, S_1, \ldots, p_M, r_M, S_M, N_1, \ldots, N_{M-1}) \\ &= \lim_{l\to\infty} b_1^b(l) = b_1^b \\ s_M(n) &= \Phi_2(p_1, r_1, S_1, \ldots, p_M, r_M, S_M, N_1, \ldots, N_{M-1}) \\ &= \lim_{l\to\infty} s_M^f(l) = s_M^f. \end{aligned} \tag{17}$$

3) Convergence: It is shown below that the convergence of Procedure 1 depends on the convergence of serial line Procedure 2. Although a formal proof of the convergence of Procedure 2 is not available, all numerical experiments in [14] and in our tests have produced convergent sequences. As a result, operator Φ_1 exhibits monotonic properties. Here, we present this fact as a hypothesis.

Hypothesis 1: Procedure 2 is convergent and operator $\Phi_1(p_1, r_1, \ldots, p_M, r_M, N_1, \ldots, N_{M-1})$ is monotonically decreasing with respect to p_1, and increasing with respect to r_1.

Let $I_s = \{k_r, k_t, rk_p, j_r r, j_r t, j_r m, j_p p, j_p m\}$, and $I_b = \{k_r r, k_r m, k_t t, k_t m, rk_p p, rk_r r, j_r, j_p\}$, we obtain the followng.

Proposition 1: Under Hypothesis 1, Procedure 1 is convergent, i.e., the following limits exist:

$$\lim_{n\to\infty} s_i(n) = s_i, \qquad i \in I_s; \quad \lim_{n\to\infty} b_i(n) = b_i, \qquad i \in I_b. \tag{18}$$

Proof: See the Appendix. ∎

Using the limits in (18), the system throughput can be evaluated as

$$\begin{aligned} \widehat{\mathrm{TP}} = \mathrm{TP}_M &\left(p_{k_r} + r_{k_r}\left(s_{k_r} + \alpha_r - \alpha_r s_{k_r}\right), r_{k_r}(1 - \alpha_r)\right. \\ &\left. \cdot (1 - s_{k_r}), S_{k_r}, \ldots, p_M, r_M, S_M, N_{k_r}, \ldots, N_{M-1}\right). \end{aligned} \tag{19}$$

4) Accuracy: The accuracy of estimate (19) is investigated numerically. Dozens of systems defined by assumptions i)–vii), with various machine and buffer parameters setting are simulated. In all the experiments we carried out, zero initial conditions for all buffers have been assumed in each run of the corresponding discrete event model. In addition, 10 000 time slots of warm up period have been carried out. The next 100 000 slots of stationary regime have been used to statistically evaluate throughput. The 95% confidence intervals for all statistical estimates have been evaluated with 20 runs. The confidence intervals throughout this paper are less than ± 0.0025. The estimates of throughput have relatively high precision with the largest discrepancy of 3.05% among all the experiments. Due to page limitation, eight of them are shown in Table I, where TP denotes the system throughput obtained from simulation and $\widehat{\mathrm{TP}}$ represents the throughput calculated by Procedure 1.

TABLE I
ACCURACY OF THROUGHPUT ESTIMATION
$(\mathrm{err}\% = (|\mathrm{TP} - \widehat{\mathrm{TP}}|/\mathrm{TP}) \cdot 100\%)$

Ex.1				
p_i: 0.22 0.34 0.32 0.27 0.35 0.44 0.21		r_i: 0.68 0.94 0.79 0.84 0.88 0.96 0.87		$\widehat{TP}$
S_i: 1.3 1.5 1.2 1.8 1.9 1.4 1.6		N_i: 4 3 3 4 2 3		0.421
p_{ri}: 0.25 0.36 0.28	p_{ti}: 0.25 0.35	p_{pi}: 0.44 0.39	α_i: 0.45	TP
r_{ri}: 0.82 0.97 0.74	r_{ti}: 0.89 0.92	r_{pi}: 0.93 0.91	0.1 0.35	0.423
S_{ri}: 1 1.3 0.9	S_{ti}: 1.1 1.4	S_{pi}: 0.9 0.8	k: 6 4 2	$err\%$
N_{ri}: 3 2 3 4	N_{ti}: 4 3 3	N_{pi}: 3 4	j: 2 5	0.33
Ex.2				
p_i: 0.1 0.12 0.1 0.13 0.12 0.1		r_i: 0.68 0.74 0.79 0.69 0.65 0.78		$\widehat{TP}$
S_i: 1.2 1.5 1.6 1.6 1.4 1.3		N_i: 3 3 4 3 4		0.861
p_{ri}: 0.13 0.12	p_{ti}: 0.12	p_{pi}: 0.11	α_i: 0.15	TP
r_{ri}: 0.76 0.75	r_{ti}: 0.65	r_{pi}: 0.81	0.1 0.8	0.860
S_{ri}: 1 1.1	S_{ti}: 1.3	S_{pi}: 1	k: 5 3 1	$err\%$
N_{ri}: 3 2 2	N_{ti}: 2 3	N_{pi}: 2 2	j: 2 4	0.17
Ex.3				
p_i: 0.01 0.02 0.03 0.02 0.01 0.01 0.02		r_i: 0.18 0.14 0.19 0.18 0.1 0.15 0.17		$\widehat{TP}$
S_i: 1.2 1.4 1.3 1.6 1.3 1.1 1.2		N_i: 2 3 2 2 3 2		0.646
p_{ri}: 0.03 0.02	p_{ti}: 0.02	p_{pi}: 0.01	α_i: 0.1	TP
r_{ri}: 0.16 0.15	r_{ti}: 0.15	r_{pi}: 0.11	0.15 0.4	0.665
S_{ri}: 0.9 0.8	S_{ti}: 1.1	S_{pi}: 0.8	k: 6 3 1	$err\%$
N_{ri}: 2 2 2	N_{ti}: 3 2	N_{pi}: 3 3	j: 2 5	2.87
Ex.4				
p_i: 0.1 0.15 0.12 0.06 0.03 0.09		r_i: 0.79 0.81 0.84 0.78 0.65 0.82		$\widehat{TP}$
S_i: 1 1.3 1.2 1.1 0.9 1.4		N_i: 2 3 2 4 3		0.695
p_{ri}: 0.09 0.07	p_{ti}: 0.1	p_{pi}: 0.05	α_i: 0.1	TP
r_{ri}: 0.8 0.85	r_{ti}: 0.75	r_{pi}: 0.9	0.1 0.1	0.717
S_{ri}: 1.1 1.3	S_{ti}: 1.2	S_{pi}: 1.5	k: 5 3 1	$err\%$
N_{ri}: 3 2 3	N_{ti}: 2 3	N_{pi}: 3 2	j: 2 4	3.05
Ex.5				
p_i: 0.04 0.08 0.12 0.08 0.04 0.04 0.08		r_i: 0.72 0.56 0.76 0.72 0.4 0.6 0.68		$\widehat{TP}$
S_i: 1.1 1.3 1.4 1.3 1.5 1.2 1.1		N_i: 2 2 2 2 2 2		0.773
p_{ri}: 0.03 0.02	p_{ti}: 0.02	p_{pi}: 0.01	α_i: 0.1	TP
r_{ri}: 0.16 0.15	r_{ti}: 0.15	r_{pi}: 0.11	0.1 0.1	0.760
S_{ri}: 1 1.2	S_{ti}: 0.8	S_{pi}: 1	k: 6 3 1	$err\%$
N_{ri}: 2 2 2	N_{ti}: 2 2	N_{pi}: 2 2	j: 2 5	1.62
Ex.6				
p_i: 0.1 0.08 0.09 0.07 0.04 0.05 0.08 0.10 0.07 0.06 0.1		r_i: 0.78 0.74 0.75 0.77 0.79 0.82 0.73 0.84 0.83 0.78 0.76		$\widehat{TP}$
S_i: 1 1.2 1.3 1.5 1.1 0.9 1.1 1.3 1.2 0.8 1.2		N_i: 3 4 2 3 5 3 2 3 4 2		0.666
p_{ri}: 0.05 0.06 0.07	p_{ti}: 0.09	p_{pi}: 0.07	α_i: 0.2	TP
r_{ri}: 0.73 0.75 0.76	r_{ti}: 0.88	r_{pi}: 0.81	0.1 0.3	0.672
S_{ri}: 1.2 0.9 1	S_{ti}: 1.2	S_{pi}: 1	k: 9 5 2	$err\%$
N_{ri}: 3 2 4 3	N_{ti}: 2 2	N_{pi}: 2 3	j: 3 7	0.83
Ex.7				
p_i: 0.02 0.04 0.05 0.03 0.02 0.04 0.01 0.03 0.05 0.02 0.04		r_i: 0.33 0.45 0.48 0.39 0.27 0.37 0.22 0.34 0.42 0.31 0.43		$\widehat{TP}$
S_i: 0.8 1.1 1.2 0.9 1 0.9 1.2 1.3 1.4 1.2 1		N_i: 3 3 2 3 4 2 3 2 3 2		0.539
p_{ri}: 0.04 0.03 0.05	p_{ti}: 0.05	p_{pi}: 0.02	α_i: 0.2	TP
r_{ri}: 0.48 0.44 0.46	r_{ti}: 0.41	r_{pi}: 0.32	0.15 0.5	0.542
S_{ri}: 1.1 0.8 0.9	S_{ti}: 1.1	S_{pi}: 0.8	k: 9 7 1	$err\%$
N_{ri}: 2 3 3 2	N_{ti}: 3 2	N_{pi}: 2 3	j: 3 8	0.44
Ex.8				
p_i: 0.06 0.12 0.18 0.15 0.07 0.11		r_i: 0.89 0.92 0.93 0.84 0.88 0.79 0.86		$\widehat{TP}$
S_i: 1.2 1.3 1.5 0.8 1.6 1.1 1.4		N_i: 3 2 4 3 4 3		0.537
p_{ri}: 0.03 0.06 0.07	p_{ti}: 0.15	p_{pi}: 0.04	α_i: 0.1	TP
r_{ri}: 0.82 0.78 0.83	r_{ti}: 0.85	r_{pi}: 0.8	0.2 0.3	0.537
S_{ri}: 0.9 1.1 1.5	S_{ti}: 1.2	S_{pi}: 0.8	k: 6 4 1	$err\%$
N_{ri}: 2 3 4 3	N_{ti}: 3 2	N_{pi}: 2 2	j: 2 5	0.06

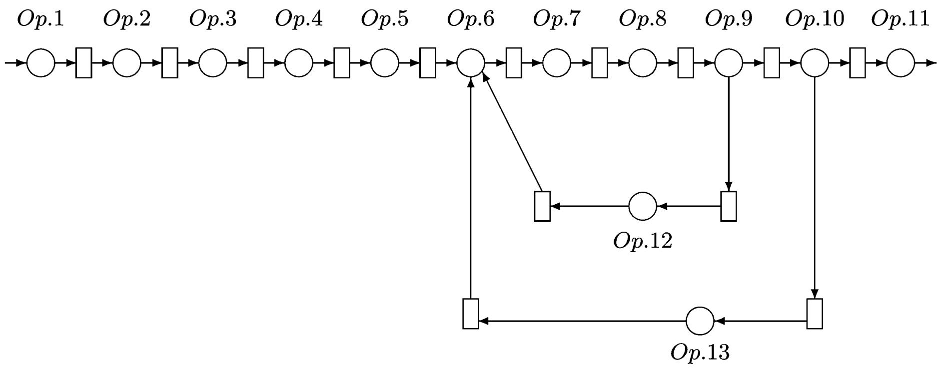

Fig. 4. Simplified system layout.

TABLE II
IDENTIFIED SYSTEM PARAMETERS

	Op.1	Op.2	Op.3	Op.4	Op.5	Op.6	Op.7
p_i	0.04	0.03	0.04	0.02	0.01	0.07	0.03
r_i	0.75	0.20	0.29	0.35	0.22	0.42	0.26
S_i	1.26	0.90	1.09	1.04	0.93	1.20	1.99
N_i	14	4	10	14	40	33	16

	Op.8	Op.9	Op.10	Op.11	Op.12	Op.13
p_i	0.15	0.02	0.02	0.02	0.08	0.04
r_i	0.20	0.44	0.32	0.33	0.17	2.52
S_i	2.00	1.60	1.68	2.06	1.26	1.60
N_i	20	52	24		22, 6	27, 7

TABLE III
CONTINUOUS IMPROVEMENTS STEP 1

	Op.1	Op.2	Op.3	Op.4	Op.5	Op.6	Op.7
S_i	1.26	0.90	1.09	1.04	0.93	1.20	1.99
E_i	1.20	0.78	0.96	0.98	0.88	1.03	1.80
$\partial\widehat{TP}/\partial S_i$	0	0	0	0	0	0.06	0

	Op.8	Op.9	Op.10	Op.11	Op.12	Op.13
S_i	2.00	1.60	1.68	2.06	1.26	1.60
E_i	1.15	1.52	1.59	1.92	0.86	1.58
$\partial\widehat{TP}/\partial S_i$	0	0	0	0	0	0

TABLE IV
CONTINUOUS IMPROVEMENTS STEP 2

	Op.1	Op.2	Op.3	Op.4	Op.5	Op.6	Op.7
S_i	1.26	0.90	1.09	1.04	0.93	1.44	1.99
E_i	1.20	0.78	0.96	0.98	0.88	1.24	1.80
$\partial\widehat{TP}/\partial S_i$	0	0.66	0.02	0	0	0	0

	Op.8	Op.9	Op.10	Op.11	Op.12	Op.13
S_i	2.00	1.60	1.68	2.06	1.26	1.60
E_i	1.15	1.52	1.59	1.92	0.86	1.58
$\partial\widehat{TP}/\partial S_i$	0	0	0	0	0	0

IV. APPLICATION

The overlapping decomposition method developed in Section III has been applied in several automotive paint shops to estimate system throughput and guide continuous improvement process. It has been shown in all applications that the throughput estimates have a good match with actual operations (with all differences less than 3%). Due to page length limitations, only one typical example of these applications is introduced in this paper.

A. Model

A simplified system layout of an automotive paint shop is shown in Fig. 4. The system consists of two loops: repair and tutone. (Compared to Fig. 1, the polishing loop is not included, in other words, $\alpha_p = 0$.) Beginning from Op. 1, jobs go through all operations 2–9. If it is a tutone job, it is sent to tutone operation through Op. 12, and then back to Op. 6. Otherwise, it will be inspected at Op. 10. Qualified jobs are prepared for general assembly. Defective jobs are repaired at Op. 13 and sent back to Op. 6 for repaint. The identified machine and buffer parameters are shown in Table II. In addition, both tutone and rework rates are $\alpha_r = \alpha_t = 0.15$.

By using Procedure 1, the system throughput is calculated as 0.74 job per unit of time. This estimate has less than 2% difference compared with actual operations. (Note that the layouts illustrated in this paper only represent the basic flow of materials. In addition, the data has been modified appropriately. They are used for demonstration only).

B. Improvements

The goal of throughput analysis of existing production lines is to continuously improve system performance. More than 10% increment of system throughput is the target in this project. One of the most important jobs in continuous improvement of production systems is to identify and eliminate system bottlenecks. A bottleneck machine is the machine that impedes system performance in the strongest manner compared to all other machines. In this study, we are interested in identifying the *speed* bottleneck, which is defined as follows.

Definition 1: Machine m_i is the speed bottleneck if

$$\frac{\partial \mathrm{TP}}{\partial S_i} > \frac{\partial \mathrm{TP}}{\partial S_j} \qquad \forall j \neq i. \tag{20}$$

Using Definition 1, we identify speed bottlenecks to improve system performance, where $(\partial\mathrm{TP}/\partial S_i)$ is calculated through $(\mathrm{TP}(S_i+\delta) - \mathrm{TP}(S_i))/\delta, 0 < \delta \ll 1$. The continuous improvement process is based on the following 4 steps.

- *Step 1.* Identify speed bottleneck as Op. 6 (Table III). In Table III and subsequent tables, values of each operation's speed, S_i, and isolated machine throughput, E_i, are also illustrated. It is shown in Table III that the speed bottleneck (Op. 6) is neither the operation with the slowest speed, nor with the smallest isolated throughput.
- *Step 2.* Increase speed of Op. 6 by 20% to 1.44 jobs per unit of time and identify Op. 2 as a new speed bottleneck (Table IV). The throughput is increased to 0.75 job per unit of time, which is a 1.31% improvement.
- *Step 3.* Increase speed of Op. 2 by 20% to 1.08 jobs per unit of time, and identify Op. 8 as a new speed bottleneck (Table V). The system throughput is increased to 0.81 job per unit of time, which is 9.46% improvement in total.

TABLE V
CONTINUOUS IMPROVEMENTS STEP 3

	Op.1	Op.2	Op.3	Op.4	Op.5	Op.6	Op.7
S_i	1.26	1.08	1.09	1.04	0.93	1.44	1.99
E_i	1.20	0.93	0.96	0.98	0.88	1.24	1.80
$\partial\widehat{TP}/\partial S_i$	0	0	0	0	0	0.01	0.01

	Op.8	Op.9	Op.10	Op.11	Op.12	Op.13
S_i	2.00	1.60	1.68	2.06	1.26	1.60
E_i	1.15	1.52	1.59	1.92	0.86	1.58
$\partial\widehat{TP}/\partial S_i$	0.28	0.03	0	0	0	0

- *Step 4.* Increase speed of Op. 8 by 20% to 1.92 jobs per unit of time. The system throughput is increased to 0.82 job per unit of time, which is 10.81% improvement in total. The desired throughput improvement is obtained.

  The above analysis has been submitted to the corresponding continuous improvement teams and they accepted these recommendations.

V. CONCLUSION

In this paper, an overlapping decomposition method is presented to approximate system throughput in automotive paint shops (i.e., systems with unequal machine speeds and multiple rework loops). The accuracy of the method has been justified with a good match by comparing with results from simulations and actual operations. In addition, we applied the method at an automotive paint shop to guide a continuous improvement project and obtained good results.

APPENDIX
PROOF OF PROPOSITION 1

To prove Proposition 1, the following two Lemmas are needed.

Lemma 1: Under Hypothesis 1, operator $\Phi_2(p_1, r_1, \ldots, p_M, r_M, N_1, \ldots, N_{M-1})$ is monotonically decreasing with respect to p_M, and decreasing with respect to r_M.

Proof: Follows from Hypothesis 1 by reversibility of production lines [16]. ■

Lemma 2: Under Hypothesis 1, in Procedure 1, if $s_i(n) > s_i(n-1), i \in I_s$, and $b_i(n) < b_i(n-1), i \in I_b$, then $s_i(n+1) > s_i(n)$ and $b_i(n+1) < b_i(n)$.

Proof: First, for line 6, if

$$\begin{aligned} s_{k_r}(n) &> s_{k_r}(n-1) \\ b_{rk_pp}(n) &< b_{rk_pp}(n-1) \\ b_{rk_pr}(n) &< b_{rk_pr}(n-1) \end{aligned}$$

from (1), we have $r'_{k_r}(n+1) < r'_{k_r}(n)$ and $r'_{rk_p}(n+1) > r'_{rk_p}(n)$. It follows from Hypothesis 1 and Lemma 1 that

$$b_{k_rr}(n+1) < b_{k_rr}(n) \quad s_{rk_p}(n+1) > s_{rk_p}(n). \tag{21}$$

Similarly, for line 7, from (2) and using (21), we have $r''_{rk_p}(n+1) < r''_{rk_p}(n)$ and $r'_{j_r}(n+1) > r''_{j_r}(n)$; then, by Hypothesis 1 and Lemma 1, it follows that

$$b_{rk_pr}(n+1) < b_{rk_pr}(n) \quad s_{j_rr}(n+1) > s_{j_rr}(n). \tag{22}$$

For lines 9 and 8, from (3) and (4), using (22) and (21), respectively, we have

$$\begin{aligned} r'_{k_t}(n+1) &< r'_{k_t}(n) \\ r''_{j_r}(n+1) &> r''_{j_r}(n) \\ r'''_{rk_p}(n+1) &< r'''_{rk_t}(n) \\ r'_{j_p}(n+1) &> r'_{j_p}(n). \end{aligned}$$

It implies that

$$\begin{aligned} b_{k_tt}(n+1) &< b_{k_tt}(n) \\ s_{j_rt}(n+1) &> s_{j_rt}(n) \\ b_{rk_pp}(n+1) &< b_{rk_pp}(n) \\ s_{j_pp}(n+1) &> s_{j_pp}(n). \end{aligned} \tag{23}$$

Next, for lines 5 and 1, from (5) and (6), using (22) and (23), it follows that $r''_{k_r}(n+1) < r''_{k_r}(n)$ and $r'''_{j_r}(n+1) > r'''_{j-1}(n)$, which leads to

$$b_{k_rm}(n+1) < b_{k_rm}(n) \quad s_{j_rm}(n+1) > s_{j_rm}(n). \tag{24}$$

Then, finally, for lines 2–4, from (7)–(9), using (21)–(24), it follows that

$$\begin{aligned} r''''_{j_r}(n+1) &< r''''_{j_r}(n) \\ r''_{k_t}(n+1) &> r''_{k_t}(n) \\ r'''_{k_t}(n+1) &< r'''_{k_t}(n) \\ r''_{j_p}(n+1) &< r''_{j_p}(n) \\ r'''_{j_p}(n+1) &< r'''_{j_p}(n) \\ r'''_{k_r}(n+1) &< r'''_{k_r}(n). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} b_{j_r}(n+1) &< b_{j_r}(n) \\ b_{k_tm}(n+1) &< b_{k_tm}(n) \\ b_{j_p}(n+1) &< b_{j_p}(n) \\ s_{k_t}(n+1) &> s_{k_t}(n) \\ s_{j_pm}(n+1) &> s_{j_pm}(n) \\ s_{k_r}(n+1) &> s_{k_r}(n). \end{aligned} \tag{25}$$

Therefore, we conclude that

$$b_i(n+1) < b_i(n), \quad i \in I_b, \quad s_i(n+1) > s_i(n), \quad i \in I_s.$$

■

Proof of Proposition 1: By induction. For $n = 0$

$$\begin{aligned} s_i(0) &= 0, \quad i = k_r, k_t \\ b_i(0) &= 1, \quad i = j_r, j_p, k_tm, rk_pp, rk_pr. \end{aligned}$$

The proof of the base case involves $n = 1, 2, 3, 4$. For $n = 1$, from Line 6, we have $r'_{k_r}(1) = r_{k_r}\alpha_r(1 - s_{k_r}(1))$ and $r'_{rk_p}(1) = 0$, which implies that

$$b_{k_rr}(1) = 1 \quad s_{rk_p}(1) = 0.$$

Analogously, for lines 9 and 8, we have

$$b_{k_tt}(1) = b_{rk_pp}(1) = 1 \quad s_{j_rt}(1) = s_{j_pp}(1) = 0.$$

For line 5, $r''_{k_r}(1) = r_{k_r}(1 - \alpha_r)$, it follows that

$$b_{k_rm}(1) < 1.$$

Similarly, for lines 1–3, we obtain

$$b_{j_r}(1) = b_{k_tm}(1) = 0 \quad s_{j_rm}(1) = s_{k_t}(1) = s_{j_pm}(1) = 0.$$

Finally, for line 4, $r'''_{j_p}(1) = r_{j_p}, r'''_{k_r}(1) = r_{k_r}(1 - \alpha_r)(1 - b_{k_rm}(1))$, and

$$s_{k_r}(1) > 0 \quad b_{j_p}(1) < 1.$$

Now, proceed with $n = 2, 3$, and 4 with similar arguments by iterating from lines 6 to 4. Finally, we obtain

$$\begin{aligned} b_{rk_pr}(4) &< 1 \\ s_{j_rr}(4) &> 0 \\ b_{k_tt}(4) &< 1 \\ s_{j_rt}(4) &> 0 \\ s_{j_rm}(4) &> 0 \end{aligned}$$

and

$$\begin{aligned} b_i(4) &< b_i(3), \qquad i = k_rr, rk_pp, k_rm, j_r, k_tm, j_p \\ s_i(4) &> s_i(3), \qquad i = rk_p, j_rr, j_pp, k_t, j_pm, k_r. \end{aligned}$$

The base case is proved. Assume now that $n > 0$

$$s_i(n) > s_k(n-1), \quad i \in I_s, \quad b_i(n) < b_i(n-1), \quad i \in I_b. \quad (26)$$

Then, from Lemma 2 ,we obtain

$$s_i(n+1) > s_k(n), \quad i \in I_s, \quad b_i(n+1) < b_i(n), \quad i \in I_b.$$

Therefore, $s_i(n)$ and $b_i(n)$ are monotonically increasing or decreasing, respectively. Since they are bounded by 0 and 1 [14], they are convergent.

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Productivity of Parallel Production Lines With Unreliable Machines and Material Handling

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***Abstract*—Using parallelism in bufferless production lines can improve productivity, with significant productivity gains achieved with crossover. However, including crossover in the line implies additional material-handling requirements that may reduce the availability of the system. This paper examines if parallel systems with crossover between the stages are more productive than parallel systems without crossover between the stages, when one considers the availability of the additional material handling required for the crossover. The minimum material-handling availability necessary for inclusion of crossover is determined for a given parallel line's configuration such that productivity can be maximized.**

***Note to Practitioners*—Two approaches in configuring parallel manufacturing lines are currently being used in industrial plants. These have been characterized as the Japanese approach of parallel independent cells of serial operations, and the European approach of a serial line with each operation being duplicated in parallel. The European approach has a productivity advantage over the Japanese approach when considering machine failures within each operation. However, the European approach requires more material handling which increases the configuration complexity and can reduce productivity. A math model is developed to determine which approach is best for a given line design when line length is defined by process planning and line balancing, and line width is determined by throughput requirements. The analysis is limited to cell configurations that do not use buffers internal to the cell.**

***Index Terms*—Availability, material handling, productivity, system analysis and design.**

I. INTRODUCTION

Configuration is an important, sometimes overlooked, aspect of the manufacturing-system design that can significantly effect its performance. Its effect has been studied by Koren *et al.* [1] who noted its impact on such parameters as reliability, productivity, quality, scalability, convertibility, and cost. For manufacturing-system design decisions involving capital expenditures, one of the most important parameters is productivity. Traditionally, system productivity is estimated from the availability of the system elements. In automated machining transfer lines, and to a lesser extent in assembly lines, productivity shortfalls due to equipment failures are customarily addressed by the inclusion

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