A THEORY ON ANISOTROPIC PIEZOTHERMOELASTIC SHELL LAMINATES WITH SENSOR/ACTUATOR APPLICATIONS

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Piezoelectric materials have become very popular in dynamic sensing, actuation and control of active structural and mechanical systems in recent years. This paper sets out to provide a comprehensive thermo-electromechanical theory of anisotropic piezoelectric thin shell laminates subjected to mechanical, electric and temperature excitations. Piezothermoelastic constitutive equations of anisotropic piezoelectric materials are defined, and governing thermo-electromechanical equations and boundary conditions are derived using Hamilton's principle. In general, the resultant forces/moments and electric displacements have three components, contributed by the elastic, electric and thermal fields, respectively; they interact in the system thermo-electromechanical equations and boundary conditions. Applications of the theory to dynamic sensing and control are also discussed. Due to the generalities of the material and geometry, the derived theory can be widely used in many piezoelectric materials, e.g., piezoceramics, piezo-polymers, etc., and also geometries, e.g., shells, plates, rings, beams, etc. Application examples are given in case studies.

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I. INTRODUCTION

In the development of intelligent structural systems, piezoelectrics are widely used as sensors and actuators for the dynamic monitoring and control of structural and mechanical systems [1]. (This new emerging area, which encompasses electromechanical systems, active materials, controls and elastic structural continua, can be called Structronics (structure-electronics) similar to the mechatronics introduced in the early 1980's.) In general, a piezoelectric material responds to force/pressure and generates a charge/voltage; this is referred to as the direct piezoelectric effect. On the other hand, the material exhibits stress/strain changes when a strong external charge/voltage is applied, and this is called the converse piezoelectric effect. In addition, the material could also respond to a temperature fluctuation and generate a charge/voltage, and this is referred to as the pyroelectric effect. It has been over 113 years since the first discovery of piezoelectric material. However, it was only relatively recent that engineers and scientists have again recognized the potential of piezoelectric materials in distributed sensors/actuators, high precision systems, micro-electromechanical systems, adaptive structural systems, etc. [1–3]. Consequently, significant efforts have been devoted to the research and development of piezoelectric technologies in recent years.

In the development of distributed sensors, actuators and thin-film devices, thin-layer piezoelectrics (either laminated, deposited or embedded) are of importance in many
applications, e.g., dynamic measurement, control, actuation, etc. Hubbard, Crawley and Baz et al. have investigated distributed sensing and control effects of piezoelectric laminated beams [4–7]. Tzou and Gadre [8] have proposed a multi-layer shell actuator theory based on an equivalent induced strain approximation. Hagood, Chung and von Flotow [9] have studied the modelling of piezoelectric actuator dynamics. Polla [10] has utilized micromachining techniques to fabricate thin-film multi-function micro sensors/actuators. Tzou, Zhong, Natori and Hollkamp have studied distributed orthogonal sensors and actuators [11, 12]. Tzou and Bao [13] have modelled thick piezoelectric laminated shell transducers. In these studies, distributed sensing and control of structures have been studied, and sensing/control effectiveness demonstrated.

Besides mechanical and electric couplings and interactions, temperature can also influence the performance of piezoelectric devices. For example, temperature variations can introduce voltage/charge generation in piezoelectric sensors. In addition, control voltages can cause temperature rise in piezoelectric actuators. Thermo-electromechanical coupling of anisotropic piezothermoelastic shell laminates is investigated in this study. Mindlin [14] has derived the governing equations of a linear piezothermoelastic medium. W. Nowacki [15] has proposed a uniqueness theorem for the solutions of piezothermoelastic differential equations, while J. P. Nowacki has discussed the influence of a temperature field on an elastic dielectric medium and a reciprocity theorem. Tzou and Howard [17] have proposed a piezothermoelastic shell vibration theory with applications to active structures. Tzou and Ye [18] have evaluated the temperature effects on distributed piezoceramic sensors and actuators in distributed sensing and control applications. It was noted that temperature can introduce the pyroelectric effect and the thermal strain effect to distributed sensors, and also thermal deflection in dynamic oscillations. To extend the initial investigations of thermo-electromechanical couplings in distributed sensors and actuators, this paper is concerned with a generic thermo-electromechanical theory of thin anisotropic piezothermoelastic shell laminates, in which the thermo-electromechanical characteristics of piezoelectric shell laminates are investigated and their sensor/actuator applications are discussed. Thermo-electromechanical equations and boundary conditions of arbitrary anisotropic piezoelectric shell laminates are derived using Hamilton’s principle. A simplification of the generic theory to a shell laminate made of a symmetrical orthorhombic piezoelectric material (mm2) is demonstrated and its sensor/actuator applications discussed. The mm2 shell lamination theory is further simplified to a single-layer piezoelectric shell with temperature fluctuations.

2. DEFINITIONS

In this section, the fundamental geometry and constitutive relations are defined, and these definitions are used in later derivations. It is assumed that a piezothermoelastic laminated composite shell is composed of $n$ anisotropic piezoelectric laminae: each of them can be used as either sensor, actuator or structural component. The shell is defined in a curvilinear co-ordinate system $(x_1, x_2, x_3)$ in which the $x_1$- and $x_2$-axes define the reference surface and $x_3$ defines the transverse axis normal to the reference surface (see Figure 1). The piezothermoelastic laminated shell is exposed to electric, mechanical and thermal fields. Note that all strain and stress components are defined in Lagrangian co-ordinates (i.e., defined in the undeformed state) such that Green strain and Kirchhoff stress components are used in later derivations.
A generic infinitesimal distance $ds$ in a shell lamina can be defined as [1, 19]

$$(ds)^2 = \sum_{i=1}^{3} (f_\alpha)^2 (dx_i)^2,$$  \hspace{1cm} (1)$$

where $f_\alpha(x_1, x_2, x_3) = A_i(1 + x_3/R_i)$, $i = 1, 2$, $f_{13}(x_1, x_2, x_3) = 1$; $A_1$ and $A_2$ are Lamé parameters; and $R_1$ and $R_2$ are the radii of curvature of the $x_1$- and $x_2$-axes on the surface defined by $x_3 = 0$ (i.e., the reference surface, not necessary the middle surface). The relations between the electric fields $E_1$, $E_2$ and $E_3$ and the electric potential $\varphi$ in the curvilinear co-ordinate system are [1]

$$\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = -\begin{pmatrix} f_{11} & 0 & 0 \\ 0 & f_{22} & 0 \\ 0 & 0 & f_{33} \end{pmatrix}^{-1} \begin{pmatrix} \partial \varphi / \partial x_1 \\ \partial \varphi / \partial x_2 \\ \partial \varphi / \partial x_3 \end{pmatrix}. \hspace{1cm} (2)$$

The linear piezothermoelastic constitutive equations of a piezothermoelastic continuum in a multiple field (a combination of mechanical, electric and thermal fields) can be expressed as [14, 18]

$$(T) = [\epsilon] (S) - [\epsilon] (E) - \{\lambda\} \theta, \hspace{0.5cm} (D) = [\epsilon] (S) + [\epsilon] (E) + [\rho] \theta,$$

$$\varphi = \{\lambda\}^T (S) + \{\rho\}^T (E) + x_3 \theta. \hspace{1cm} (3a-c)$$

where $\{T\}$, $\{S\}$, $\{E\}$ and $\{D\}$ denote the stress, strain, electric field and electric displacement vectors, respectively; $\varphi$ is the thermal entropy density; $\theta$ is the temperature rise ($\theta = \Theta - \Theta_0$, where $\Theta$ is the absolute temperature and $\Theta_0$ is the temperature of the natural state in which stresses and strains are zero); $[\epsilon]$, $[\epsilon]$ and $[\epsilon]$ denote the elastic stiffness coefficient, piezoelectric coefficient and dielectric permittivity matrices, respectively; $\{\lambda\}$ is the stress-temperature coefficient vector; $\{\rho\}$ is the pyroelectric coefficient vector; and $x_3$ is a material constant ($x_3 = \rho c_p / \Theta_0$, where $c_p$ is the specific heat at constant volume). $[\cdot]^T$ and $\{\cdot\}^T$ are matrix and vector transpose, respectively. For a complete
anisotropic piezothermoelastic material, and constitutive equations of the $k$th lamina can be written as

$$
\begin{pmatrix}
\bar{T}_{1k} \\
\bar{T}_{2k} \\
\bar{T}_{3k} \\
\bar{T}_{4k} \\
\bar{T}_{5k} \\
\bar{T}_{6k}
\end{pmatrix} =
\begin{pmatrix}
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} & \varepsilon_{14} & \varepsilon_{15} & \varepsilon_{16} \\
\varepsilon_{22} & \varepsilon_{23} & \varepsilon_{24} & \varepsilon_{25} & \varepsilon_{26} & \\
\varepsilon_{33} & \varepsilon_{34} & \varepsilon_{35} & \varepsilon_{36} & \\
\varepsilon_{44} & \varepsilon_{45} & \varepsilon_{46} & \\
\varepsilon_{55} & \varepsilon_{56} & \\
\varepsilon_{66}
\end{pmatrix}_k
\begin{pmatrix}
\bar{S}_1 \\
\bar{S}_2 \\
\bar{S}_3 \\
\bar{S}_4 \\
\bar{S}_5 \\
\bar{S}_6
\end{pmatrix}_k
- \begin{pmatrix}
\bar{E}_1 \\
\bar{E}_2 \\
\bar{E}_3 \\
\bar{E}_4 \\
\bar{E}_5 \\
\bar{E}_6
\end{pmatrix}_k
\begin{pmatrix}
\bar{\tau}_1 \\
\bar{\tau}_2 \\
\bar{\tau}_3 \\
\bar{\tau}_4 \\
\bar{\tau}_5 \\
\bar{\tau}_6
\end{pmatrix}_k
\begin{pmatrix}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\theta_5 \\
\theta_6
\end{pmatrix}_k
\tag{4a}
$$

$$
\begin{pmatrix}
\bar{D}_{1k} \\
\bar{D}_{2k} \\
\bar{D}_{3k}
\end{pmatrix} =
\begin{pmatrix}
\bar{\varepsilon}_{11} & \bar{\varepsilon}_{12} & \bar{\varepsilon}_{13} & \bar{\varepsilon}_{14} & \bar{\varepsilon}_{15} & \bar{\varepsilon}_{16} \\
\bar{\varepsilon}_{21} & \bar{\varepsilon}_{22} & \bar{\varepsilon}_{23} & \bar{\varepsilon}_{24} & \bar{\varepsilon}_{25} & \bar{\varepsilon}_{26} \\
\bar{\varepsilon}_{31} & \bar{\varepsilon}_{32} & \bar{\varepsilon}_{33} & \bar{\varepsilon}_{34} & \bar{\varepsilon}_{35} & \bar{\varepsilon}_{36} \\
\bar{\varepsilon}_{41} & \bar{\varepsilon}_{42} & \bar{\varepsilon}_{43} & \bar{\varepsilon}_{44} & \bar{\varepsilon}_{45} & \bar{\varepsilon}_{46} \\
\bar{\varepsilon}_{51} & \bar{\varepsilon}_{52} & \bar{\varepsilon}_{53} & \bar{\varepsilon}_{54} & \bar{\varepsilon}_{55} & \bar{\varepsilon}_{56} \\
\bar{\varepsilon}_{61} & \bar{\varepsilon}_{62} & \bar{\varepsilon}_{63} & \bar{\varepsilon}_{64} & \bar{\varepsilon}_{65} & \bar{\varepsilon}_{66}
\end{pmatrix}_k
\begin{pmatrix}
\bar{\varepsilon}_1 \\
\bar{\varepsilon}_2 \\
\bar{\varepsilon}_3 \\
\bar{\varepsilon}_4 \\
\bar{\varepsilon}_5 \\
\bar{\varepsilon}_6
\end{pmatrix}_k
+ \begin{pmatrix}
\bar{\varphi}_1 \\
\bar{\varphi}_2 \\
\bar{\varphi}_3
\end{pmatrix}_k
\begin{pmatrix}
\theta_1 \\
\theta_2 \\
\theta_3
\end{pmatrix}_k
\tag{4b}
$$

Note that $\{\varepsilon\}, \{\varphi\}, \{\varepsilon\}, \{\varepsilon\}, \{\varphi\}, \{\varepsilon\}, \{\varphi\}$ and $\{\varphi\}$ are defined with respect to the $k$th layer's principal material co-ordinate axes $x_1, x_2$ and $x_3; \{\varepsilon\}, \{\varphi\}, \{\varepsilon\}, \{\varphi\}$ (terms with a bar) are defined with respect to the arbitrary curvilinear axes $x_1, x_2$ and $x_3$ [20]. The subscript "k" denotes the $k$th layer parameters. (Note that in generic unsymmetric triclinic piezoelectric materials, $\varepsilon_{13} = \varepsilon_{21}, \varepsilon_{23} = \varepsilon_{12}$ and $\varepsilon_{13} = \varepsilon_{23}$). The generic anisotropic piezothermoelastic material is used in deriving the generic piezothermoelastic system equations, and their simplifications to other piezothermoelastic materials, e.g., mm2 and mm6, are then discussed.
3. HAMILTON'S PRINCIPLE

Hamilton's principle is used to derive the system thermo-electromechanical equations and boundary conditions of the piezothermoelastic shell continuum. Hamilton's principle assumes that the energy variation over an arbitrary period of time is zero, and the equation is expressed as

$$\delta \int_{t_0}^{t_f} (\mathcal{H} \mathcal{l} - \Pi) \, dt = 0,$$

where \( \mathcal{H} \)

$$\mathcal{H} = \int_V \left[ \frac{1}{2} \rho \dot{U}_j \dot{U}_j \right] \, dV.$$

\( \Pi \) is the total potential energy (including electric enthalpy, thermal energy, work done by surface tractions and electric potential energy [17]):

$$\Pi = \int_V \left[ \mathcal{H}(S_1, E_j, \Theta) + \Theta \nabla \right] \, dV - \int_S \left[ t_j \dot{U}_j - Q_j \phi \right] \, dS.$$

Note that \( \rho \) is the mass density, \( \mathcal{H} \) is the electric enthalpy, \( t_j \) is the surface traction in the \( \alpha_j \) direction, \( Q_j \) is the surface electric charge, \( \phi \) is the electrical potential, \( V \) and \( S \) are the volume and surface of the piezothermoelastic continuum, respectively, and \( U_j \) and \( \dot{U}_j \) are the displacement and velocity vectors. Substituting all energy expressions into Hamilton's equation gives

$$\delta \int_{t_0}^{t_f} \int_V \left[ \frac{1}{2} \rho \dot{U}_j \dot{U}_j - \mathcal{H}(S_1, E_j, \Theta) - \Theta \nabla \right] \, dV \, dt + \int_{t_0}^{t_f} \int_S \left[ t_j \delta \dot{U}_j - Q_j \delta \phi \right] \, dS \, dt = 0.$$

The electric enthalpy \( \mathcal{H} \) is a function of strains, electric fields and temperatures:

$$\mathcal{H} = \int \left( \frac{1}{2} \sigma_{ij} \varepsilon_{ij}^{(e)} - \frac{1}{2} \sigma_{ii} \varepsilon_{ii}^{(e)} + \sigma_{ij} \varepsilon_{ij}^{(e)} + \frac{1}{2} \varepsilon_{ii}^{(e)} \varepsilon_{ii}^{(e)} + \frac{1}{2} \sigma_{ij} \varepsilon_{ij}^{(e)} - \frac{1}{2} \sigma_{ij} \varepsilon_{ij}^{(e)} + \sigma_{ij} \varepsilon_{ij}^{(e)} - \frac{1}{2} \sigma_{ii} \varepsilon_{ii}^{(e)} \right) \, dV.$$

It is assumed that \( n \) layers of piezothermoelastic shell laminae are laminated together to become a single composite piezothermoelastic laminated system. Each shell lamina could be either a sensor, an actuator or a structural element in the laminated system. Then, the total electric enthalpy \( \mathcal{H} \) of the laminated shell becomes

$$\mathcal{H} = \sum_{k=1}^{n} (\mathcal{H})_k = \frac{1}{2} \sum_{k=1}^{n} \left( \varepsilon_{ii}^{(e)} \varepsilon_{ii}^{(e)} + \sigma_{ij} \varepsilon_{ij}^{(e)} - \frac{1}{2} \sigma_{ii} \varepsilon_{ii}^{(e)} \right) \, dV - \frac{1}{2} \sum_{k=1}^{n} \left( \varepsilon_{ii}^{(e)} \varepsilon_{ii}^{(e)} + \sigma_{ij} \varepsilon_{ij}^{(e)} - \frac{1}{2} \sigma_{ii} \varepsilon_{ii}^{(e)} \right) \, dV.$$

Recall that \( \varepsilon_{ij} \), \( \sigma_{ij} \), \( \varepsilon_{ii}^{(e)} \), \( \sigma_{ij}^{(e)} \), \( \varepsilon_{ii}^{(e)} \), \( \sigma_{ii}^{(e)} \) and \( \sigma_{ij}^{(e)} \) are defined with respect to the \( k \)-th layer's principal material co-ordinates \( x_1, x_2 \) and \( x_3 \), and \( \varepsilon_{ij}^{(e)} \), \( \sigma_{ij}^{(e)} \), \( \varepsilon_{ii}^{(e)} \), \( \sigma_{ii}^{(e)} \), \( \varepsilon_{ii}^{(e)} \), \( \sigma_{ij}^{(e)} \) and \( \sigma_{ii}^{(e)} \) are defined with respect to the arbitrary curvilinear axes \( x_1, x_2 \) and \( x_3 \). Including all shell laminae, one can rewrite Hamilton's equation for the laminated shell composite as

$$\int_{t_0}^{t_f} \int_{S_1} \int_{V_1} \left( \frac{1}{2} \rho \varepsilon_{ij}^{(e)} \right) \, dV \, dt - \int_{t_0}^{t_f} \int_{S_1} \int_{V_1} \left( \varepsilon_{ii}^{(e)} \varepsilon_{ii}^{(e)} + \sigma_{ij} \varepsilon_{ij}^{(e)} - \frac{1}{2} \sigma_{ii} \varepsilon_{ii}^{(e)} \right) \, dV \, dt + \int_{t_0}^{t_f} \int_{S_1} \int_{V_1} \left( t_j \delta \dot{U}_j - Q_j \delta \phi \right) \, dS \, dt = 0.$$
This equation will be used to derive the system thermo-electromechanical equations and boundary conditions.

4. THIN PIEZOTHERMOELASTIC SHELL LAMINATES

It is assumed that the composite piezothermoelastic shell laminate (with a constant thickness $h$) is made of $n$ anisotropic piezothermoelastic laminae (each has a constant thickness of $h_i$) perfectly bonded together. The in-plane electric fields $\vec{E}_1$ and $\vec{E}_2$ are neglected and only the transverse electric field $\vec{E}_3$ is considered. In this section, the strain-displacement relations and stress resultants (forces or moments per unit length) of the laminated anisotropic piezothermoelastic shell are defined.

The laminated shell is still considered to be thin, so that the effects from the transverse shear deformations and rotatory inertias can be neglected. It is also assumed that the displacements $U_i$ and $U_j$ vary linearly through the laminated shell thickness, and that the transverse displacement $U_j$ is independent of the thickness. Thus, the displacement fields are assumed as [1, 19]:

$$U_i(x_1, x_2, x_3, t) = u_i(x_1, x_2, t) + x_3 \beta_i(x_1, x_2, t),$$

$$U_j(x_1, x_2, x_3, t) = u_j(x_1, x_2, t) + x_3 \beta_j(x_1, x_2, t),$$

$$U_{ij}(x_1, x_2, x_3, t) = u_{ij}(x_1, x_2, t).$$  \hspace{1cm} (12a–c)

where $u_i$, $u_j$, and $u_{\alpha}$ are the displacement components of a point of the reference surface along the $\alpha_1$- and $\alpha_3$- and $\alpha_2$-axes; $\beta_1$ and $\beta_2$ represent angles of rotation in the positive sense of the $\alpha_1$- and $\alpha_2$-axes, respectively. The displacements are independent variables; however, the rotational angles are dependent variables which can be derived from the thin shell assumptions. For a thin shell, the transverse normal strain $\bar{S}_3$ and shear strains $\bar{S}_i$ and $\bar{S}_j$ are negligible, i.e.,

$$\bar{S}_3 = 0, \quad \bar{S}_i = \beta_i - \frac{u_i}{R_1} + \frac{1}{A_1} \frac{\partial u_3}{\partial x_1} = 0, \quad \bar{S}_j = \beta_j - \frac{u_j}{R_2} + \frac{1}{A_2} \frac{\partial u_3}{\partial x_2} = 0. \hspace{1cm} (13a–c)$$

Thus, the rotational angles for the thin laminated shell are defined as

$$\beta_1 = \frac{u_1}{R_1} - \frac{1}{A_1} \frac{\partial u_3}{\partial x_1}, \quad \beta_2 = \frac{u_2}{R_2} - \frac{1}{A_2} \frac{\partial u_3}{\partial x_2}. \hspace{1cm} (14a, b)$$

In general, $x_3/R_i \ll 1$ and $x_3/R_j \ll 1$. Thus, the ratio of the finite distance to the radius of curvature is negligible, and so $f_{11} \simeq A_1$ and $f_{22} \simeq A_2$. Accordingly, the linear strain-displacement relations can be written as a summation of membrane strains and bending strains:

$$\begin{pmatrix} \bar{S}_1 \\ \bar{S}_2 \\ \bar{S}_3 \end{pmatrix} = \begin{pmatrix} \bar{s}_1 \\ \bar{s}_2 \\ \bar{s}_3 \end{pmatrix} + x_3 \begin{pmatrix} \bar{\kappa}_1 \\ \bar{\kappa}_2 \\ \bar{\kappa}_3 \end{pmatrix}, \hspace{1cm} (15)$$

where

$$\bar{s}_1 = \frac{1}{A_1} \frac{\partial u_1}{\partial x_1} - \frac{u_3}{A_1} \frac{\partial A_1}{\partial x_1} - \frac{u_1}{R_1}, \quad \bar{s}_2 = \frac{1}{A_2} \frac{\partial u_2}{\partial x_2} - \frac{u_3}{A_2} \frac{\partial A_2}{\partial x_2} + \frac{u_2}{R_2}, \quad \bar{s}_3 = \frac{1}{A_1} \frac{\partial u_3}{\partial x_1} + \frac{u_1}{A_1} \frac{\partial A_1}{\partial x_1} + \frac{u_3}{R_1}.$$

\hspace{1cm} (16a, b)
\[ s_0^o = \frac{1}{A_1} \frac{\partial u_1}{\partial x_2} + \frac{1}{A_1} \frac{\partial u_2}{\partial x_1} - \frac{u_1}{A_1} \frac{\partial A_1}{\partial x_2} - \frac{u_2}{A_1} \frac{\partial A_2}{\partial x_1}, \]  
\[ \kappa_1 = \frac{1}{A_1} \frac{\partial \beta_1}{\partial x_1} + \frac{\beta_2}{A_1 A_2} \frac{\partial A_1}{\partial x_2}, \quad \kappa_2 = \frac{1}{A_1} \frac{\partial \beta_2}{\partial x_1} + \frac{\beta_1}{A_1 A_2} \frac{\partial A_2}{\partial x_1}, \]  
\[ \kappa_5 = \frac{1}{A_2} \frac{\partial \beta_1}{\partial x_2} + \frac{\beta_2}{A_1 A_2} \frac{\partial A_1}{\partial x_1}, \quad \kappa_6 = \frac{1}{A_2} \frac{\partial \beta_2}{\partial x_1} + \frac{\beta_1}{A_1 A_2} \frac{\partial A_2}{\partial x_1}. \]

\( s_0^o \), \( s_0^e \) and \( s_0^e \) are the membrane strains, while \( \kappa_1 \), \( \kappa_2 \) and \( \kappa_6 \) are the bending strains (the change of curvatures on the reference surface). The stress resultants \( N_{ij} \) (or the forces per unit length of the reference surface arc) and the stress couples \( M_{ij} \) (or the moments per unit length of the reference surface arc) are contributed by mechanical, electric and thermal components:

\[ \begin{bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{bmatrix} = \int \begin{bmatrix} T_1 \\ T_2 \\ T_5 \end{bmatrix} dx_5 = \begin{bmatrix} \langle T_1 \rangle \\ \langle T_2 \rangle \\ \langle T_5 \rangle \end{bmatrix} - \begin{bmatrix} \langle T_1 \rangle^m \\ \langle T_2 \rangle^m \\ \langle T_5 \rangle^m \end{bmatrix} - \begin{bmatrix} \langle T_1 \rangle^e \\ \langle T_2 \rangle^e \\ \langle T_5 \rangle^e \end{bmatrix} - \begin{bmatrix} \langle T_1 \rangle^\theta \\ \langle T_2 \rangle^\theta \\ \langle T_5 \rangle^\theta \end{bmatrix} + \begin{bmatrix} \langle T_1 \rangle^t \end{bmatrix}, \]

\[ \begin{bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{bmatrix} = \int \begin{bmatrix} T_1 \\ T_2 \\ T_6 \end{bmatrix} dx_5 = \begin{bmatrix} \langle T_1 \rangle \\ \langle T_2 \rangle \\ \langle T_6 \rangle \end{bmatrix} - \begin{bmatrix} \langle T_1 \rangle^m \\ \langle T_2 \rangle^m \\ \langle T_6 \rangle^m \end{bmatrix} - \begin{bmatrix} \langle T_1 \rangle^e \\ \langle T_2 \rangle^e \\ \langle T_6 \rangle^e \end{bmatrix} - \begin{bmatrix} \langle T_1 \rangle^\theta \\ \langle T_2 \rangle^\theta \\ \langle T_6 \rangle^\theta \end{bmatrix} + \begin{bmatrix} \langle T_1 \rangle^t \end{bmatrix}, \]

The transverse shear forces are

\[ \begin{bmatrix} Q_{21} \\ Q_{31} \end{bmatrix} = \int \begin{bmatrix} T_1 \\ T_3 \end{bmatrix} dx_3 = \begin{bmatrix} \langle T_1 \rangle \\ \langle T_3 \rangle \end{bmatrix} - \begin{bmatrix} \langle T_1 \rangle^m \\ \langle T_3 \rangle^m \end{bmatrix} - \begin{bmatrix} \langle T_1 \rangle^e \\ \langle T_3 \rangle^e \end{bmatrix} - \begin{bmatrix} \langle T_1 \rangle^\theta \\ \langle T_3 \rangle^\theta \end{bmatrix} + \begin{bmatrix} \langle T_1 \rangle^t \end{bmatrix}, \]

Here for convenience, \( \langle \cdots \rangle \) denotes the thickness integration, i.e.,

\[ \langle \cdots \rangle = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \cdots dx_3. \]

\( \{ T \}_{k}^m = [e]_{k} \{ S \}, \quad \{ T \}_{k}^e = [e]_{k} \{ E \}, \quad \{ T \}_{k}^\theta = [e]_{k} \theta. \) The superscript \( m \) denotes the mechanically induced components, \( e \) the electrically induced components, and \( \theta \) the...
thermally induced components. The resultant forces and moments (mechanical, electric and thermal) can be further organized in a compact matrix form:

\[
\begin{pmatrix}
N_{11} \\
N_{22} \\
N_{12} \\
M_{11} \\
M_{22} \\
M_{12}
\end{pmatrix}
= \begin{pmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & A_{26} & B_{21} & B_{22} & B_{26} \\
A_{16} & A_{26} & A_{66} & B_{61} & B_{62} & B_{66} \\
B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\
B_{12} & B_{22} & B_{26} & D_{21} & D_{22} & D_{26} \\
B_{16} & B_{26} & B_{66} & D_{61} & D_{62} & D_{66}
\end{pmatrix}
\begin{pmatrix}
\bar{e}_1^2 \\
\bar{e}_2^2 \\
\bar{e}_6^2 \\
\bar{e}_1 \\
\bar{e}_2 \\
\bar{e}_6
\end{pmatrix}
- \begin{pmatrix}
N_{11}' \\
N_{22}' \\
N_{12}' \\
M_{11}' \\
M_{22}' \\
M_{12}'
\end{pmatrix}
, \tag{18}
\]

where \(A_{ij}, B_{ij}\) and \(D_{ij}\) are the extensional, coupling and bending stiffnesses, defined by

\[
A_{ij} = \langle (\bar{e}_{ij})_k \rangle = \sum_{k=1}^{n} \langle (\bar{e}_{ij})_k (x_{3k} - x_{3k-1}) \rangle, \tag{19a}
\]

\[
B_{ij} = \langle (\bar{e}_{ij})_k x_3 \rangle = \frac{1}{2} \sum_{k=1}^{n} \langle (\bar{e}_{ij})_k (x_{1k}^2 - x_{1k-1}^2) \rangle, \tag{19b}
\]

\[
D_{ij} = \langle (\bar{e}_{ij})_k x_3^2 \rangle = \frac{1}{2} \sum_{k=1}^{n} \langle (\bar{e}_{ij})_k (x_{1k}^4 - x_{1k-1}^4) \rangle. \tag{19c}
\]

Also, the inertia force terms in three axial directions can be written as

\[
\int_4 \rho_k \bar{U}_i \, dx_3 = \langle \rho_k (\bar{u}_i + x_3 \bar{\beta}_i) \rangle = \mathcal{A} \bar{u}_i + \mathcal{B} \bar{\beta}_i, \quad i = 1, 2, \tag{20a}
\]

\[
\int_4 \rho_k \bar{U}_i \, dx_3 = \langle \rho_k (\bar{u}_i + x_3 \bar{\beta}_i) x_3 \rangle = \mathcal{B} \bar{u}_i + \mathcal{D} \bar{\beta}_i, \quad i = 1, 2, \tag{20b}
\]

\[
\int_4 \rho_k \bar{U}_3 \, dx_3 = \langle \rho_k \bar{u}_3 \rangle = \mathcal{A} \bar{u}_3, \tag{20c}
\]

where the coefficients \(\mathcal{A}, \mathcal{B}\) and \(\mathcal{D}\) are defined by all \(n\) laminae in the shell laminate:

\[
\mathcal{A} = \sum_{k=1}^{n} \rho_k (x_{3k} - x_{3k-1}), \quad \mathcal{B} = \frac{1}{2} \sum_{k=1}^{n} \rho_k (x_{1k}^2 - x_{1k-1}^2), \quad \mathcal{D} = \frac{1}{2} \sum_{k=1}^{n} \rho_k (x_{1k}^4 - x_{1k-1}^4). \tag{21a-c}
\]

Although it is assumed that the transverse shear deflections can be neglected, the integrated effect of the transverse shear stresses \(Q_{ij}\) and \(Q_{23}\) are not neglected. These transverse shear forces will be defined by the resultant equations \([1, 19]\). For a symmetric laminated shell with the middle surface as the reference surface, the coupling stiffnesses \(B_{ij}\) and \(\mathcal{B}\) are zero. In addition, on the basis of Bernoulli–Euler theory, the effects of the rotatory inertias \(\mathcal{A} \bar{\beta}_i\) and \(\mathcal{D} \bar{\beta}_i\) can also be neglected in thin shells.

5. PIEZOTHERMOELASTIC EQUATIONS AND BOUNDARY CONDITIONS

Substituting all energy terms into Hamilton’s equation and carrying out all variations, one can derive the thermo-electromechanical equations, the charge equation of electro-
statics, and the boundary conditions of the thin piezothermoelastic shell laminate. The piezothermoelastic equations in three axial directions are

\[
\frac{\partial (N_{11} A_1)}{\partial z_1} - N_{22} \frac{\partial A_2}{\partial z_1} + \frac{\partial (N_{22} A_1)}{\partial z_1} + N_{12} \frac{\partial A_1}{\partial z_2} + Q_{13} \frac{A_1 A_2}{R_1} = A_1 A_2 \rho \delta_{11}, \tag{22a}
\]

\[
\frac{\partial (N_{12} A_2)}{\partial z_1} + N_{21} \frac{\partial A_1}{\partial z_1} + \frac{\partial (N_{21} A_1)}{\partial z_1} - N_{11} \frac{\partial A_1}{\partial z_2} + Q_{23} \frac{A_1 A_2}{R_2} = A_1 A_2 \rho \delta_{12}, \tag{22b}
\]

\[
\frac{\partial (Q_{13} A_2)}{\partial z_1} + \frac{\partial (Q_{23} A_1)}{\partial z_2} + A_1 A_2 \sum_{k=1}^{\infty} \tilde{F}_k \frac{\partial \tilde{u}_k}{\partial z_k} - A_1 A_2 \left( \frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} \right) = A_1 A_2 \rho \delta_{13}, \tag{22c}
\]

where transverse shear stress resultants \(Q_{13}\) and \(Q_{23}\) are defined by

\[
\frac{\partial (M_{11} A_2)}{\partial z_1} - M_{22} \frac{\partial A_2}{\partial z_1} + \frac{\partial (M_{22} A_1)}{\partial z_1} + M_{12} \frac{\partial A_1}{\partial z_2} - A_1 A_2 Q_{11} = 0, \tag{23a}
\]

\[
\frac{\partial (M_{12} A_2)}{\partial z_1} + M_{21} \frac{\partial A_1}{\partial z_1} + \frac{\partial (M_{21} A_1)}{\partial z_1} - M_{11} \frac{\partial A_1}{\partial z_2} - A_1 A_2 Q_{22} = 0, \tag{23b}
\]

and where \(\rho\) is defined as a weighted average density for the multi-layered shell, i.e.

\[
\rho = \frac{1}{h} \sum_{k=1}^{\infty} \rho_k h_k.
\]

The transverse stress resultant \(A_1 A_2 \sum_{k=1}^{\infty} \tilde{F}_k \frac{\partial \tilde{u}_k}{\partial z_k}\) can be neglected because it is usually small in thin shells or the thickening effect can be ignored. Substituting stress resultants \(N_{11}, N_{22}, N_{12}, M_{11}, M_{22}\) and \(M_{12}\) into the above equations and simplifying them accordingly, one can derive the system equations in terms of middle-surface displacements \(u_x, u_y, u_z\). The resulting expressions are lengthy, and they are not presented here. It is observed that there are three thermo-electromechanical equations governing the mechanical motion in the \(x_1, x_2, x_3\) directions respectively, while \(Q_{13}\) and \(Q_{23}\) are defined by two rotational equations. Note that all elastic, electric and thermal related terms are included in the force and moment expressions. These electric terms can be used, in conjunction with control algorithms, as control forces/moments counteracting mechanical and temperature induced vibrations in distributed structural control of shells [1, 12, 17].

The charge equation of electrostatics is

\[
\frac{\partial (\Bar{D}_{12} \delta_{13})}{\partial z_1} + \frac{\partial (\Bar{D}_{22} \delta_{11})}{\partial z_2} + \frac{\partial (\Bar{D}_{33} \delta_{12})}{\partial z_3} = 0, \tag{24}
\]

where the electric displacements \(\Bar{D}_k\) are as defined previously. Note that the in-plane electric fields \(\Bar{E}_1\) and \(\Bar{E}_2\) are assumed to be zero in the thin piezothermoelastic laminated shell. The charge equation shows that, if the variation of the electric potential is not zero, the divergence of electric displacement equals zero within the piezoelectric body of each layer. This charge equation of electrostatics can be used in sensor applications [1, 11]. The detailed thermo-electromechanical coupling of piezoelectric sensors is discussed in a case study presented later.
Mechanical boundary conditions can be derived from the variational equation, and they can be either force/moment boundary conditions or displacement/rotation boundary conditions denoted by a superscript asterisk (*).

**Boundaries defined by** \( x_1 \) **are as follows:**

\[
N_{11} = N_{11}^* \quad \text{or} \quad u_1 = u_1^*, \quad (25a)
\]

\[
M_{11} = M_{11}^* \quad \text{or} \quad \beta_1 = \beta_1^*, \quad (25b)
\]

\[
N_{12} + \frac{M_{12}}{R_2} = V_{12}^* \quad \text{or} \quad u_2 = u_2^*, \quad (25c)
\]

\[
Q_{13} + \frac{1}{A_2} \frac{\partial M_{12}}{\partial x_2} = V_{13}^* \quad \text{or} \quad u_3 = u_3^*, \quad (25d)
\]

**Boundaries defined by** \( x_2 \) **are as follows:**

\[
N_{22} = N_{22}^* \quad \text{or} \quad u_2 = u_2^*, \quad (26a)
\]

\[
M_{22} = M_{22}^* \quad \text{or} \quad \beta_2 = \beta_2^*, \quad (26b)
\]

\[
N_{21} + \frac{M_{21}}{R_1} = V_{21}^* \quad \text{or} \quad u_1 = u_1^*, \quad (26c)
\]

\[
Q_{23} + \frac{1}{A_1} \frac{\partial M_{21}}{\partial x_1} = V_{23}^* \quad \text{or} \quad u_3 = u_3^*, \quad (26d)
\]

The boundary shear stress resultants \( V_{ij}^* \) are defined as

\[
V_{11}^* = N_{11}^* + \frac{M_{12}}{R_2}, \quad V_{22}^* = N_{22}^* + \frac{M_{21}}{R_1}, \quad (27a, b)
\]

\[
V_{12}^* = Q_{12}^* + \frac{1}{A_2} \frac{\partial M_{12}}{\partial x_2}, \quad V_{23}^* = Q_{23}^* + \frac{1}{A_1} \frac{\partial M_{21}}{\partial x_1}. \quad (27c, d)
\]

Usually only either the force boundary conditions or the displacement boundary conditions are selected for a given physical boundary condition [1, 19]. Electric boundary conditions on the transverse surface \( S_k \), are

\[
D_{3k} + Q_{3k}^* = 0. \quad (28)
\]

Note that it is assumed that each layer is electrically isolated. Thus, the electric displacement on the surface of the \( k \)th layer equals the external charge density on that surface.

Note that these thermo-electromechanical equations and boundary conditions are derived based on anisotropic piezothermoelastic materials. They can be used for generic piezothermoelastic laminated transducers, laminated composite shell structures, etc. Simplifications of the theory to other materials and/or specific geometries can easily be accomplished once all geometrical and/or material parameters are defined, and these are demonstrated in case studies.

6. CASE STUDIES

In this section, generic thermo-electromechanical equations and boundary conditions of the thin anisotropic piezothermoelastic laminated shell are simplified to a number of
6.1. CASE 1: CLASS mm2 SYMMETRIC PIEZOELECTRIC MATERIALS

It is assumed that the thin laminated shell is made of $n$ piezothermoelastic laminae; each lamina is a symmetric orthorhombic material of class mm2 corresponding to the principal material axes $x_1$, $x_2$ and $x_3$, respectively [1]. The constitutive equations of the $k$th layer, referring to the axes $x_1$, $x_2$ and $x_3$, are

\[
\begin{bmatrix}
T_1 \\
T_2 \\
T_3 \\
0 \\
0
\end{bmatrix}_k = \begin{bmatrix}
c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\
0 & c_{22} & c_{23} & 0 & 0 & 0 \\
0 & 0 & c_{33} & 0 & 0 & 0 \\
\text{symm.} & c_{44} & 0 & 0 & 0 & 0 \\
\end{bmatrix}_k \begin{bmatrix}
S_1 \\
S_2 \\
S_3 \\
S_4 \\
S_5 \\
S_6
\end{bmatrix}_k
\]

\[
- \begin{bmatrix}
0 & 0 & e_{31} \\
0 & 0 & e_{32} \\
0 & 0 & e_{33} \\
est & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}_k \begin{bmatrix}
E_1 \\
E_2 \\
E_3
\end{bmatrix}_k - \begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
0 \\
0
\end{bmatrix} \theta, \quad (29a)
\]

\[
\begin{bmatrix}
D_1 \\
D_2 \\
D_3
\end{bmatrix}_k = \begin{bmatrix}
0 & 0 & 0 & e_{13} & 0 \\
0 & 0 & 0 & 0 & 0 \\
est & est & 0 & 0 & 0
\end{bmatrix}_k \begin{bmatrix}
S_1 \\
S_2 \\
S_3 \\
S_4 \\
S_5 \\
S_6
\end{bmatrix}_k
\]

\[
+ \begin{bmatrix}
\epsilon_{11} & 0 & 0 \\
0 & \epsilon_{22} & 0 \\
0 & 0 & \epsilon_{33}
\end{bmatrix}_k \begin{bmatrix}
E_1 \\
E_2 \\
E_3
\end{bmatrix}_k + \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}_k \theta. \quad (29b)
\]
(Note that \( \varepsilon_{34} \neq \varepsilon_{15} \) for mm2 polyvinylidene fluoride polymer, and \( \varepsilon_{34} = \varepsilon_{15} \) for mm6 piezoceramics.) The constitutive equations referring to the shell curvilinear axes \( \alpha_1, \alpha_2, \) and \( \alpha_3 \) become

\[
\begin{bmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
T_5 \\
T_6
\end{bmatrix} =
\begin{bmatrix}
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} & 0 & 0 & \varepsilon_{16} \\
\varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} & 0 & 0 & \varepsilon_{26} \\
\varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} & 0 & 0 & \varepsilon_{36} \\
\varepsilon_{41} & \varepsilon_{42} & \varepsilon_{43} & 0 & 0 & \varepsilon_{46} \\
\varepsilon_{51} & \varepsilon_{52} & \varepsilon_{53} & 0 & 0 & \varepsilon_{56} \\
\varepsilon_{61} & \varepsilon_{62} & \varepsilon_{63} & 0 & 0 & \varepsilon_{66}
\end{bmatrix}
\begin{bmatrix}
S_1 \\
S_2 \\
S_3 \\
S_4 \\
S_5 \\
S_6
\end{bmatrix}
- \begin{bmatrix}
0 \\
0 \\
0 \\
\varepsilon_{14} \\
\varepsilon_{24} \\
\varepsilon_{34}
\end{bmatrix}
\begin{bmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
T_5 \\
T_6
\end{bmatrix}.
\]

\[
\times
\begin{bmatrix}
\frac{E_1}{E_1} \\
\frac{E_2}{E_2} \\
\frac{E_3}{E_3}
\end{bmatrix}
- \begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
0 \\
0 \\
\lambda_4
\end{bmatrix}
\begin{bmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
T_5 \\
T_6
\end{bmatrix}
- \begin{bmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
T_5 \\
T_6
\end{bmatrix}
\begin{bmatrix}
S_1 \\
S_2 \\
S_3 \\
S_4 \\
S_5 \\
S_6
\end{bmatrix},
\]

\[
\begin{bmatrix}
\tilde{D}_1 \\
\tilde{D}_2 \\
\tilde{D}_3
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 & \varepsilon_{14} & \varepsilon_{15} & 0 \\
0 & 0 & 0 & \varepsilon_{24} & \varepsilon_{25} & 0 \\
\varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} & 0 & 0 & \varepsilon_{36}
\end{bmatrix}
\begin{bmatrix}
S_1 \\
S_2 \\
S_3 \\
S_4 \\
S_5 \\
S_6
\end{bmatrix}
+ \begin{bmatrix}
\varepsilon_{11} & \varepsilon_{12} & 0 \\
\varepsilon_{12} & \varepsilon_{12} & 0 \\
0 & 0 & \varepsilon_{33}
\end{bmatrix}
\begin{bmatrix}
\frac{E_1}{E_1} \\
\frac{E_2}{E_2} \\
\frac{E_3}{E_3}
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
\tilde{\beta}_1
\end{bmatrix} \theta.
\]

Note that the in-plane electric fields \( E_1 \) and \( E_2 \) are neglected in thin shells. The relationships between the transformed quantities \( \tilde{\varepsilon}_{ij} \) and the original \( \varepsilon_{ij} \) are [20]:

\[
\begin{align*}
\varepsilon_{11} &= c_{11} c^4 + 2(c_{12} + 2c_{66})c^2 s^2 + c_{22} s^4, \\
\varepsilon_{12} &= (c_{11} + c_{22} - 4c_{66})c^2 s^2 + c_{12}(c^4 + s^4), \\
\varepsilon_{13} &= c_{13} c^2 + c_{23} s^2, \\
\varepsilon_{16} &= -c_{22} c s^3 + c_{11} c^3 s - (c_{12} + 2c_{66})c s(c^2 - s^2), \\
\varepsilon_{22} &= c_{11} s^4 + 2(c_{12} + 2c_{66})c^2 s^2 + c_{22} c^4, \\
\varepsilon_{23} &= -c_{22} c^3 s + c_{11} c^3 s - (c_{12} + 2c_{66})c s(c^2 - s^2), \\
\varepsilon_{33} &= c_{13}, \\
\varepsilon_{36} &= c_{23}, \\
\varepsilon_{44} &= c_{22} c^3 s + c_{11} c^3 s - (c_{12} + 2c_{66})c^2 s, \\
\varepsilon_{45} &= (c_{13} - c_{23})c s, \\
\varepsilon_{55} &= c_{13} c^2 + c_{33} s^2, \\
\varepsilon_{56} &= (c_{11} + c_{22} - 2c_{12})c^2 s^2 + c_{66}(c^2 - s^2)^2.
\end{align*}
\]
where $c = \cos \beta$ and $s = \sin \beta$, and $\beta$ is an orientation angle from the $x_i$-axis to the $\alpha_j$-axis. The elastic constants $c_{ij}$ can be defined by Young’s moduli $Y_{ij}$, Poisson’s ratios $\mu_{ij}$, and shear moduli $G_{ij}$ [20]:

\[
\begin{align*}
c_{11} &= Y_{11}(1 - \mu_{13}\mu_{23})/\Delta, \\
c_{22} &= Y_{22}(1 - \mu_{12}\mu_{33})/\Delta, \\
c_{33} &= Y_{33}(1 - \mu_{23}\mu_{12})/\Delta, \\
c_{44} &= G_{23}, \\
c_{55} &= G_{13}, \\
c_{66} &= G_{12}, \\
c_{12} &= Y_{12}(\mu_{23}\mu_{13})/\Delta = Y_{22}(\mu_{13}\mu_{23})/\Delta \\
c_{13} &= Y_{11}(\mu_{13}\mu_{23})/\Delta = Y_{33}(\mu_{13}\mu_{23})/\Delta, \\
c_{23} &= Y_{22}(\mu_{13}\mu_{23})/\Delta = Y_{33}(\mu_{13}\mu_{23})/\Delta.
\end{align*}
\]

(32)

where $\Delta = 1 - \mu_{12}\mu_{21} - \mu_{23}\mu_{32} - \mu_{13}\mu_{31} - 2\mu_{11}\mu_{22}\mu_{33}$.

Similarly, using $c = \cos \beta$ and $s = \sin \beta$, one can rewrite the piezoelectric constants, dielectric constants, stress–temperature coefficients, and pyroelectric constants as

\[
\begin{align*}
\tilde{e}_{14} &= sc(e_{15} - e_{24}), \\
\tilde{e}_{13} &= c^2e_{15} + s^2e_{24}, \\
\tilde{e}_{24} &= s^2e_{15} + c^2e_{24}, \\
\tilde{e}_{33} &= sc(e_{11} - e_{22}), \\
\tilde{e}_{31} &= c^2e_{11} + s^2e_{22}, \\
\tilde{e}_{32} &= s^2e_{11} + c^2e_{22}, \\
\tilde{e}_{33} &= e_{33}, \\
\tilde{e}_{36} &= sc(e_{11} - e_{22}), \\
\tilde{e}_{34} &= c^2\lambda_1 + s^2\lambda_2, \\
\tilde{e}_{43} &= s^2\lambda_1 + c^2\lambda_2, \\
\tilde{e}_{53} &= \lambda_3, \\
\tilde{e}_{63} &= sc(\lambda_1 - \lambda_2), \\
\tilde{\rho}_3 &= \rho_3.
\end{align*}
\]

(33)

The transverse electric field of the $k$th layer becomes

\[
\tilde{E}_{3k} = \frac{1}{\tilde{\xi}_{3k}} \tilde{D}_{3k} - \frac{1}{\tilde{\xi}_{3k}} (\tilde{e}_{31k} \tilde{S}_1 + \tilde{e}_{32k} \tilde{S}_2 + \tilde{e}_{36k} \tilde{S}_6) - \frac{1}{\tilde{\xi}_{3k}} \tilde{\rho}_3 \tilde{T}
\]

\[
= \tilde{E}_{3k}^e + \tilde{E}_{3k}^d + \tilde{E}_{3k}^t,
\]

(34)

where the superscripts $e$, $d$ and $\theta$ denote the components contributed by the converse, direct and pyroelectric effects, respectively. Note that the electric field is contributed by (1) electric displacement, (2) elastic strains and (3) temperature. In the case in which temperature is negligible, the electric field equations becomes the standard piezoelectric equation. On the other hand, in the absence of electric displacement and elastic strain, the electric field is solely contributed by the temperature—that is the pyroelectric effect, which is used in thermal imaging applications, e.g., the infrared camera. In dynamic sensing applications, the electric displacement is usually absent; the sensor signal is contributed by only the (dynamic) strains but also the temperature variations. Accordingly, if this sensor signal is used in feedback control, it should be understood that the signal has a temperature induced component which has nothing to do with the true dynamic state. This temperature induced component needs to be carefully handled in order precisely to execute the vibration control [18].

The thermo-electromechanical equations of the mm2 piezothermoelastic laminated shell in three axial directions can then be derived in the same form as those of the anisotropic piezothermoelastic shell laminates in equations (22) and (23). However, the definitions of resultant forces and moments are different, and these forces and moments are derived later. The mass density is the weighted mass density of all layers, denoted as

\[
\rho = \frac{1}{\tilde{h}_{i=1}^n \rho_i \tilde{h}_i},
\]

(35)
The transverse shell stress resultant

\[ A_1 A_2 \sum_{k=1}^{n} \overline{T}_{nk} \overline{E}_{nk} \]

in the transverse thermo-electromechanical equation is induced by the converse effect, and this effect can be neglected in thin shell configurations or the thickening effect is negligible. For a symmetric laminated shell made of \( n \) layers of symmetrical orthorhombic materials (class mm2 with respect to the axes \( x_1, x_2 \) and \( x_3 \)), the force and moment resultants are defined as a summation of mechanical, electric and temperature induced components:

\[
N_{11} = N_{11}^\eta - N_{11}^\zeta - N_{11}^\zeta = (A_{11} \overline{\sigma}_{11}^b + A_{12} \overline{\sigma}_{12}^b + A_{16} \overline{\sigma}_{16}^b) - \langle \overline{e}_{31} \overline{E}_{31} \rangle - \langle \overline{\lambda}_{11} \overline{\theta} \rangle,
\]

\[
N_{22} = N_{22}^\eta - N_{22}^\zeta - N_{22}^\zeta = (A_{12} \overline{\sigma}_{12}^b + A_{22} \overline{\sigma}_{22}^b + A_{26} \overline{\sigma}_{26}^b) - \langle \overline{e}_{32} \overline{E}_{32} \rangle - \langle \overline{\lambda}_{22} \overline{\theta} \rangle,
\]

\[
N_{12} = N_{12}^\eta - N_{12}^\zeta - N_{12}^\zeta = (A_{16} \overline{\sigma}_{16}^b + A_{66} \overline{\sigma}_{66}^b + A_{66} \overline{\sigma}_{66}^b) - \langle \overline{e}_{36} \overline{E}_{36} \rangle - \langle \overline{\lambda}_{12} \overline{\theta} \rangle,
\]

\[
M_{11} = M_{11}^\eta - M_{11}^\zeta - M_{11}^\zeta = (D_{11} \overline{\kappa}_{11} + D_{12} \overline{\kappa}_{12} + D_{16} \overline{\kappa}_{16}) - \langle \overline{e}_{31} \overline{E}_{31} \rangle - \langle \overline{\lambda}_{11} \overline{\theta} \rangle, \tag{37a}
\]

\[
M_{22} = M_{22}^\eta - M_{22}^\zeta - M_{22}^\zeta = (D_{12} \overline{\kappa}_{12} + D_{22} \overline{\kappa}_{22} + D_{26} \overline{\kappa}_{26}) - \langle \overline{e}_{32} \overline{E}_{32} \rangle - \langle \overline{\lambda}_{22} \overline{\theta} \rangle, \tag{37b}
\]

\[
M_{12} = M_{12}^\eta - M_{12}^\zeta - M_{12}^\zeta = (D_{16} \overline{\kappa}_{16} + D_{66} \overline{\kappa}_{66} + D_{66} \overline{\kappa}_{66}) - \langle \overline{e}_{36} \overline{E}_{36} \rangle - \langle \overline{\lambda}_{12} \overline{\theta} \rangle, \tag{37c}
\]

where \( \overline{e}_{31} \overline{E}_{31}, \overline{e}_{32} \overline{E}_{32} \) and \( \overline{e}_{36} \overline{E}_{36} \) have a value only for the piezoelectric layers. The thermo-electromechanical equations can be further defined in displacements \( u_{i}(x_{1}, x_{2}, t) \) \( u_{i}(x_{1}, x_{2}, t) \) and \( u_{i}(x_{1}, x_{2}, t) \) and electric and thermal constants. It can be observed that the resultant forces and moments essentially have three components: one elastic component, one electric component and one temperature component. In vibration control, the electric component is usually set up to reflect the dynamic state of structures and it is used to attenuate or counteract the structural vibrations. When temperature also fluctuates, in addition to the elastic vibrations, the electric component needs to counteract both the mechanical (elastic) vibration and the temperature induced oscillation [18, 21].

The mechanical boundary conditions for the \( x_{1} \) and \( x_{2} \) surfaces can be simplified to

\[
N_{11}^\eta - N_{11}^\zeta - N_{11}^\zeta = N_{11}^\zeta \quad \text{or} \quad u_{1} = u_{1}^{*}, \tag{38a}
\]

\[
M_{11}^\eta - M_{11}^\zeta - M_{11}^\zeta = M_{11}^\zeta \quad \text{or} \quad \beta_{1} = \beta_{1}^{*}, \tag{38b}
\]

\[
(N_{12}^\eta - N_{12}^\zeta - N_{12}^\zeta) + (M_{12}^\eta - M_{12}^\zeta - M_{12}^\zeta) R_{2} = V_{12}^* \quad \text{or} \quad u_{2} = u_{2}^{*}, \tag{38c}
\]

\[
Q_{12} + \frac{1}{A_{2}} \frac{\partial (M_{12}^\eta - M_{12}^\zeta - M_{12}^\zeta)}{\partial x_{2}} = V_{12}^{*} \quad \text{or} \quad u_{1} = u_{1}^{*}, \tag{38d}
\]

and

\[
N_{22}^\eta - N_{22}^\zeta - N_{22}^\zeta = N_{22}^\zeta \quad \text{or} \quad u_{2} = u_{2}^{*}, \tag{39a}
\]

\[
M_{22}^\eta - M_{22}^\zeta - M_{22}^\zeta = M_{22}^\zeta \quad \text{or} \quad \beta_{2} = \beta_{2}^{*}, \tag{39b}
\]

\[
(N_{21}^\eta - N_{21}^\zeta - N_{21}^\zeta) + (M_{21}^\eta - M_{21}^\zeta - M_{21}^\zeta) R_{1} = V_{21}^{*} \quad \text{or} \quad u_{1} = u_{1}^{*}, \tag{39c}
\]

\[
Q_{21} + \frac{1}{A_{1}} \frac{\partial (M_{21}^\eta - M_{21}^\zeta - M_{21}^\zeta)}{\partial x_{1}} = V_{21}^{*} \quad \text{or} \quad u_{2} = u_{2}^{*}. \tag{39d}
\]
The charge equation of electrostatics becomes
\[
\frac{\partial (\hat{D}_{3c}f_{1e}/22)}{\partial \varepsilon_3} = \frac{\partial \left[ \tilde{e}_{31} \tilde{S}_{11} + \tilde{e}_{32} \tilde{S}_{22} + \tilde{e}_{36} \tilde{S}_{66} + \tilde{e}_{33} \tilde{E}_3 + \tilde{e}_{31} \theta \right]_k A_1 A_2 (1 + z_1/R_1) (1 + z_3/R_3)}{\varepsilon_3} = 0.
\]
(40)

The charge equations of electrostatics can be used to estimate the output signals of a piezoelectric laminated shell in sensor applications [1, 11]. The electric boundary condition becomes
\[
(\tilde{e}_{31} \tilde{S}_{11} + \tilde{e}_{32} \tilde{S}_{22} + \tilde{e}_{36} \tilde{S}_{66} + \tilde{e}_{33} \tilde{E}_3 + \tilde{e}_{31} \theta)_k + Q^* = 0.
\]
(41)

It is noted that the combined elastic, electric and temperature effect is equal to the surface electric charge on the surface of the laminate.

6.2. CASE 2: A SINGLE-LAYER SHELL (mm2 PIEZOTHERMOELASTIC SHELL)

In a single-layer piezothermoelastic shell (n = 1, \( h_1 = h \)), the middle (neutral) surface is taken as the reference surface. Thus, the (force) constants are simplified to
\[
A_i = \tilde{e}_i h, \quad B_i = 0, \quad D_i = (h^3/12)\tilde{e}_i, \quad \mathcal{A} = \rho h, \quad \mathcal{B} = 0, \quad \mathcal{D} = (h^3/12)\rho.
\]
(42)

Note that the couplings (\( B_i \) and \( \mathcal{D} \)) between the membrane and bending components are zero. Mechanical and electric force and moment resultants are also simplified to
\[
N^*_{11} = h(\tilde{e}_{11} \tilde{S}_{11} + \tilde{e}_{12} \tilde{S}_{22} + \tilde{e}_{16} \tilde{S}_{66}), \quad M^*_{11} = (h^3/12)(\tilde{e}_{11} \tilde{E}_3 + \tilde{e}_{12} \tilde{E}_3 + \tilde{e}_{16} \tilde{E}_3),
\]
\[
N^*_{12} = h(\tilde{e}_{12} \tilde{S}_{11} + \tilde{e}_{22} \tilde{S}_{22} + \tilde{e}_{26} \tilde{S}_{66}), \quad M^*_{12} = (h^3/12)(\tilde{e}_{12} \tilde{E}_3 + \tilde{e}_{22} \tilde{E}_3 + \tilde{e}_{26} \tilde{E}_3),
\]
\[
N^*_{16} = h(\tilde{e}_{16} \tilde{S}_{11} + \tilde{e}_{66} \tilde{S}_{66}), \quad M^*_{16} = (h^3/12)(\tilde{e}_{16} \tilde{E}_3 + \tilde{e}_{66} \tilde{E}_3).
\]

\[
N^*_{11} = \int_{\alpha_1} \tilde{e}_{31} \tilde{E}_3 \, dx_3 = \int_{\alpha_1} \tilde{e}_{31} \tilde{E}_3 \, dx_3 + \int_{\alpha_1} \tilde{e}_{31} \tilde{E}_3 \, dx_3 + \int_{\alpha_1} \tilde{e}_{31} \tilde{E}_3 \, dx_3
\]
\[
= N^*_{11} + N^*_{12} + N^*_{16}.
\]

\[
N^*_{12} = \int_{\alpha_2} \tilde{e}_{32} \tilde{E}_3 \, dx_3 = \int_{\alpha_2} \tilde{e}_{32} \tilde{E}_3 \, dx_3 + \int_{\alpha_2} \tilde{e}_{32} \tilde{E}_3 \, dx_3 + \int_{\alpha_2} \tilde{e}_{32} \tilde{E}_3 \, dx_3
\]
\[
= N^*_{12} + N^*_{16}.
\]

\[
N^*_{16} = \int_{\alpha_6} \tilde{e}_{36} \tilde{E}_3 \, dx_3 = \int_{\alpha_6} \tilde{e}_{36} \tilde{E}_3 \, dx_3 + \int_{\alpha_6} \tilde{e}_{36} \tilde{E}_3 \, dx_3 + \int_{\alpha_6} \tilde{e}_{36} \tilde{E}_3 \, dx_3
\]
\[
= N^*_{16} + N^*_{12}.
\]

\[
M^*_{11} = \int \tilde{e}_{31} \tilde{E}_3 \, x_3 \, dx_3 = \int \tilde{e}_{31} \tilde{E}_3 \, x_3 \, dx_3 + \int \tilde{e}_{31} \tilde{E}_3 \, x_3 \, dx_3 + \int \tilde{e}_{31} \tilde{E}_3 \, x_3 \, dx_3
\]
\[
= M^*_{11} + M^*_{12} + M^*_{16}.
\]

\[
M^*_{12} = \int \tilde{e}_{32} \tilde{E}_3 \, x_3 \, dx_3 = \int \tilde{e}_{32} \tilde{E}_3 \, x_3 \, dx_3 + \int \tilde{e}_{32} \tilde{E}_3 \, x_3 \, dx_3 + \int \tilde{e}_{32} \tilde{E}_3 \, x_3 \, dx_3
\]
\[
= M^*_{12} + M^*_{16}.
\]
\[
M^\theta_{12} = \int \varepsilon_{36} E^\theta_{13} x_3 \, d\alpha_3 = \int \varepsilon_{36} E_{13} x_3 \, d\alpha_3 + \int \varepsilon_{36} E_{13} x_3 \, d\alpha_3 + \int \varepsilon_{36} E^\theta_{13} x_3 \, d\alpha_3 \\
= M^\theta_{12} + M^\theta_{12} + M^\theta_{12}, \tag{43}
\]

where \( E^\theta_{13} \), \( E^\theta_{12} \) and \( E^\theta_{12} \) are defined in the first case; there are, respectively, contributed by the convexe, direct and pyroelectric effects. The temperature induced force and moment components are

\[
N^\theta_{11} = \int \lambda_{1} \theta \, dx_1, \quad M^\theta_{11} = \int \lambda_{1} \theta x_1 \, d\alpha_3, \\
N^\theta_{22} = \int \lambda_{2} \theta \, dx_2, \quad M^\theta_{22} = \int \lambda_{2} \theta x_2 \, d\alpha_3, \\
N^\theta_{12} = \int \lambda_{6} \theta \, dx_3, \quad M^\theta_{12} = \int \lambda_{6} \theta x_3 \, d\alpha_3. \tag{44}
\]

6.2.1. Zero orientation angle

If the principal co-ordinate axes \( x_1 \), \( x_2 \) and \( x_3 \) coincide with the curvilinear co-ordinate axes \( x_1 \), \( x_2 \) and \( x_3 \), i.e., the orientation angle \( \beta = 0 \), the forces and moments can be further simplified to

\[
N^\theta_{11} = h(c_{11} s^2 \theta + c_{12} s^2 \theta), \quad M^\theta_{11} = (h^2/12)(c_{11} \kappa_1 + c_{12} \kappa_2), \\
N^\theta_{22} = h(c_{12} s^2 \theta + c_{22} s^2 \theta), \quad M^\theta_{22} = (h^2/12)(c_{12} \kappa_1 + c_{22} \kappa_2), \\
N^\theta_{12} = h c_{16} s^2 \theta, \quad M^\theta_{12} = (h^2/12)c_{16} \kappa_5, \\
N^\theta_{11} = \int e_{31} E_{13} \, dx_3 = \int e_{31} (E^\theta_{13} + E^\theta_{13} + E^\theta_{13}) \, dx_3 = N^\theta_{11} + N^\theta_{11} + N^\theta_{11}, \\
N^\theta_{22} = \int e_{32} E_{13} \, dx_3 = \int e_{32} (E^\theta_{13} + E^\theta_{13} + E^\theta_{13}) \, dx_3 = N^\theta_{22} + N^\theta_{22} + N^\theta_{22}, \\
N^\theta_{12} = 0, \\
M^\theta_{11} = \int e_{31} E_{12} x_3 \, dx_3 = \int e_{31} (E^\theta_{13} + E^\theta_{13} + E^\theta_{13}) x_3 \, dx_3 = M^\theta_{11} + M^\theta_{11} + M^\theta_{11}, \\
M^\theta_{22} = \int e_{32} E_{12} x_3 \, dx_3 = \int e_{32} (E^\theta_{13} + E^\theta_{13} + E^\theta_{13}) x_3 \, dx_3 = M^\theta_{22} + M^\theta_{22} + M^\theta_{22}, \\
M^\theta_{12} = 0, \\
N^\theta_{11} = \int \lambda_{1} \theta \, dx_1, \quad M^\theta_{11} = \int \lambda_{1} \theta x_1 \, d\alpha_3, \\
N^\theta_{22} = \int \lambda_{2} \theta \, dx_2, \quad M^\theta_{22} = \int \lambda_{2} \theta x_2 \, d\alpha_3, \\
N^\theta_{12} = 0, \quad M^\theta_{12} = 0, \tag{45}
\]

where the elastic constants are

\[
c_{11} = Y_{11} (1 - \mu_{12} \mu_{13}), \quad c_{22} = Y_{22}/(1 - \mu_{12} \mu_{13}), \\
c_{12} = c_{21} = \mu_{12} Y_{22} (1 - \mu_{12} \mu_{13}) = \mu_{12} Y_{11}/(1 - \mu_{12} \mu_{11}), \quad c_{16} = G_{12}. \tag{46}
\]
and where $Y_{11}$ and $Y_{22}$ are Young’s moduli; $\mu_{12}$ and $\mu_{21}$ are Poisson ratios; and $G_{12}$ is the shear modulus. Note that the effect of orientation angle is removed and these force/moment and elastic constants are explicitly defined.

6.2.2. PVDF shell

If the shell is made of piezoelectric polyvinylidene fluoride (PVDF) polymer (i.e., $Y_{11} = Y_{22} = Y, G_{11} = G, \mu_{12} = \mu_{21} = \mu$), then the forces and moments become

$$
N_{11}^m = K(s_{11}^m + \mu s_{22}^m), \quad M_{11}^m = D(\kappa_1 + \mu \kappa_2),
$$

$$
N_{22}^m = K(\mu s_{11}^m + s_{22}^m), \quad M_{22}^m = D(\mu \kappa_1 + \kappa_2),
$$

$$
N_{12}^m = \frac{K(1 - \mu)}{2} s_{12}^m, \quad M_{12}^m = \frac{D(1 - \mu)}{2} \kappa_6,
$$

where $K$ is the membrane stiffness, $K = Yh/(1 - \mu^2)$; $D$ is the bending stiffness, $D = Yh^3/12(1 - \mu^2)$. These forces and moments are identical to those directly derived by using Hamilton’s principle [1].

6.2.3. Temperature changes

(1) Constant temperature. If the single-layer shell undergoes a uniformly distributed temperature rise $\theta = \text{constant}$, the electric field $E_3$ can be defined as a summation of electric, mechanical and temperature effects:

$$
E_3 = E_3^e + E_3^m + E_3^\theta = \frac{1}{\varepsilon_{33}} Q_3 - \frac{1}{\varepsilon_{33}} (e_{31} S_1 + e_{32} S_2) - \frac{1}{\varepsilon_{33}} p_3 \theta.
$$

(48)

Accordingly, the force and moment resultants can be simply defined as

$$
N_{11}^t = N_{11}^e + N_{11}^m + N_{11}^\theta = -\frac{e_{31}}{\varepsilon_{33}} h Q_3 - \frac{e_{31}}{\varepsilon_{33}} h (e_{31} s_{11}^m + e_{32} s_{22}^m) - \frac{e_{31}}{\varepsilon_{33}} p_3 \theta h,
$$

$$
N_{22}^t = N_{22}^e + N_{22}^m + N_{22}^\theta = -\frac{e_{32}}{\varepsilon_{33}} h Q_3 - \frac{e_{32}}{\varepsilon_{33}} h (e_{31} s_{11}^m + e_{32} s_{22}^m) - \frac{e_{32}}{\varepsilon_{33}} p_3 \theta h,
$$

$$
N_{12}^t = 0,
$$

$$
M_{11}^t = M_{11}^e = -\frac{h^2 e_{31}}{12 \varepsilon_{33}} (e_{31} \kappa_1 + e_{32} \kappa_2), \quad M_{22}^t = M_{22}^e = -\frac{h^2 e_{32}}{12 \varepsilon_{33}} (e_{31} \kappa_1 + e_{32} \kappa_2),
$$

$$
M_{12}^t = 0.
$$

(49)

Note that since the temperature is uniformly distributed, there are only temperature induced in-plane membrane forces and no bending moments. These force and moment resultants are identical to those directly derived from Hamilton’s principle [17].

(2) Linear temperature variation. If the single-layer shell is subjected to a linear distributed temperature rise $\theta$ through its thickness, i.e.,

$$
\theta(x_i) = \frac{(\theta_i - \theta_h)}{h} x_i + \frac{(\theta_i + \theta_h)}{2},
$$

(50)
where $\theta_t$ and $\theta_b$, respectively, denote the temperature rises at the top and bottom surfaces of the shell, the force and moment resultants become

$$N'_{11} = N'_{11} + N'_{11} = -\frac{e_{31}}{\epsilon_{33}} h Q^1_x - \frac{e_{31}}{\epsilon_{33}} h (e_{11} s_1^2 + e_{12} s_1^2) - \frac{e_{31}}{\epsilon_{33}} p_1 h \frac{\theta_t + \theta_b}{2},$$

$$N'_{22} = N'_{22} + N'_{22} = -\frac{e_{32}}{\epsilon_{33}} h Q^2_x - \frac{e_{32}}{\epsilon_{33}} h (e_{12} s_2^2 + e_{13} s_2^2) - \frac{e_{32}}{\epsilon_{33}} p_2 h \frac{\theta_t + \theta_b}{2},$$

$$N'_{12} = 0,$$

$$M'_{11} = M'_{11} + M'_{11} = -\frac{h^2}{12 \epsilon_{33}} (e_{11} \kappa_1 + e_{12} \kappa_2) - \frac{h^2}{12 \epsilon_{33}} p_3 (\theta_t - \theta_b).$$

$$M'_{22} = M'_{22} + M'_{22} = -\frac{h^2}{12 \epsilon_{33}} (e_{12} \kappa_1 + e_{13} \kappa_2) - \frac{h^2}{12 \epsilon_{33}} p_3 (\theta_t - \theta_b).$$

$$M'_{12} = 0,$$

$$N''_{11} = \hat{\lambda}_1 h \frac{(\theta_t + \theta_b)}{2}, \quad M''_{11} = \frac{h^2}{12} \hat{\lambda}_1 (\theta_t - \theta_b),$$

$$N''_{22} = \hat{\lambda}_2 h \frac{(\theta_t + \theta_b)}{2}, \quad M''_{22} = \frac{h^2}{12} \hat{\lambda}_2 (\theta_t - \theta_b),$$

$$N''_{12} = 0, \quad M''_{12} = 0.$$ (51)

If $\theta_t$ equals $\theta_b$, the linear temperature rise case becomes the uniform temperature rise case. Note that as long as force and moment resultants are defined for a special case, the system thermo-electromechanical equations can be defined accordingly. It is assumed that in this study all material coefficients are temperature independent.

7. CONCLUDING REMARKS

Thin piezoelectric layers are widely used as distributed sensors and actuators in intelligent structural systems and thin film electromechanical devices. Accordingly, there is a need to investigate the detailed thermo-electromechanical characteristics of piezoelectric laminates. There are a number of piezoelectric materials that have different elastic and piezoelectric material matrices. The arbitrary anisotropic piezoelectric material is the most generic piezoelectric material which has full elastic and piezoelectric matrices. Depending on material symmetry, the elastic and piezoelectric matrices can be simplified to account for various piezoelectric materials [1]. On the other hand, the deep thin shell defined in a tri-orthogonal curvilinear system can easily be simplified to apply to a large number of shell and non-shell continua [1, 19]. Combination of the material and geometrical generalities provides the versatility of the piezothermoelastic shell lamination theory present in this paper. (Note that elastic properties of electrodes or rigid adhesives can be accommodated; however, viscoelastic properties of bonding materials cannot be accommodated in this theory.) System thermo-electromechanical equations and boundary conditions of a generic thin piezothermoelastic shell laminate were derived and their thermo-electromechanical characteristics discussed.

On the basis of the derived system thermo-electromechanical equations and boundary conditions, it was observed that the elastic and electric components are closely coupled.
with the temperature-induced component, and all three components are important to the overall electromechanical and dynamic behavior of the laminated shell. In general, the charge equation is used in sensor applications [1, 11], in which the signal is contributed by elastic strains (the direct piezoelectric effect) and temperature (the pyroelectric effect and thermal strain effect), while the electric displacement is usually neglected. The electric forces and moments in the dynamic equations can be used to control the shell dynamics [1, 12, 18, 21]; these electric control forces and moments, in conjunction with appropriate control algorithms, are essential to counteract both the mechanical and thermal induced oscillations. The generic piezothermoelastic shell lamination theory can serve as a design and analysis tool when new piezothermoelastic laminated composites are designed and fabricated. With appropriate material and geometry properties specified, the dynamic performance and control effectiveness of piezothermoelastic laminates can be evaluated. These detailed thermo-electromechanical interactions of specific piezothermoelastic continua made of various piezoelectric materials are to be reported on in the near future.

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REFERENCES

APPENDIX: NOMENCLATURE

\(A_1, A_2\) Lamé parameters
\(A_{ij}, B_i, D_{ij}\) extensional, coupling and bending stiffness
\(A, \mathcal{A}, \mathcal{D}\) coefficients defined by all \(n\) laminae
\(c = \cos \beta\) constant
\(c_i\) specific heat at constant volume
\(c_{ij}\) elastic constants
\([c]\) elastic stiffness coefficient matrix
\(D = Y h^3 (12(1 - \mu^2))\) bending stiffness for a thin shell
\(d\) generic infinitesimal distance
\(\{D\}\) electric displacement vector
\(d\) thermal entropy density
\([e]\) piezoelectric coefficient matrix
\(E_{1i}, E_{1}, E_{i}\) electric fields
\(\{E\}\) electric field vector
\(f_i(x_1, x_2, x_3) = A_i(1 + x_3/R)\) \(i = 1, 2; f_3(x_1, x_2, x_3) = 1\)
\(G_{ij}\) shear moduli
\(h\) thickness
\(h_k\) thickness of \(k\)th lamina
\(\mathcal{X}\) electric enthalpy
\(\mathcal{X}\) kinetic energy
\(K = Y h (1 - \mu^2)\) membrane stiffness
\(\kappa_1, \kappa_2, \kappa_3\) bending strains
\(\text{mm}, \text{mm}\) mm. piezoelectric material has planes of symmetry in the \(x_1\)- and \(x_2\)-axes; 2 denotes that the \(x_3\)-axis is a two-fold rotational axis; 6 denotes that the \(x_3\)-axis is a six-fold rotational axis
\(M_i\) moment or stress couples
\(N_i\) membrane force or stress resultant
\(\mathcal{P}_i\) pyroelectric coefficient vector
\(Q_i\) surface electric charge
\(Q_1, Q_2\) transverse shear stresses
\(R_1, R_2\) radii of curvature in the \(x_1\)- and \(x_2\)-axes
\(s = \sin \beta\) constant
\( \mathbf{\varepsilon}_1, \mathbf{\varepsilon}_2, \mathbf{\varepsilon}_3 \) membrane strains
\( S \) surface of the piezothermoelastic continuum
\( \{S\} \) strain vectors
\( t \) surface traction in the \( z \) direction
\( \{T\} \) stress vector
\( u_x, u_y, u_z \) displacements on the reference surface
\( U_i, U_j \) displacement and velocity vectors
\( V \) volume of the piezothermoelastic continuum
\( V^* \) boundary shear stress resultants
\( Y_{ii} \) Young's moduli
\( z_{x_i} \) curvilinear co-ordinates, \( x_i \) and \( z \) for the reference surface and \( z \) for the normal axis
\( \beta_1, \beta_2 \) rotation angles
\( \beta \) lamina orientation angle between the \( x_i \)-axis and the \( z_i \)-axis
\( A \) = 1 - \( \mu_{12} \mu_{23} - \mu_{23} \mu_{31} - \mu_{31} \mu_{12} \), constant
\( [\varepsilon] \) dielectric permittivity matrix
\( \varphi \) electric potential
\( \{x\} \) stress-temperature coefficient vector
\( \mu \) Poisson ratios
\( \Pi \) total potential energy
\( \rho \) mass density
\( \Theta \) = \( \Theta - \Theta_0 \), temperature rise
\( \Theta_0 \) temperature rises at top and bottom surfaces respectively
\( \Theta \) absolute temperature
\( \Theta_0 \) temperature of natural state where stresses and strains are zero
\( \langle \cdots \rangle \) thickness integration, i.e., \( \langle \cdots \rangle = \sum_{i-1}^{i} \mu_{ii} (\cdots) dx \)
\( [\mathbf{A}, \{x\}] \) matrix and vector transpose
\( \{x\} \) properties defined with respect to the \( k \)th layer's principal material co-ordinate axes
\( x_1, x_2 \) and \( x_3 \)
\( \{x\} \) (terms with a headed bar) properties defined with respect to arbitrary curvilinear axes
\( x_1, x_2 \) and \( x_3 \)

**Superscripts**
- \( * \) boundary conditions
- \( e \) electric (converse) component
- \( d \) direct effect component
- \( \theta \) pyroelectric effect component
- \( m \) mechanically induced component
- \( e \) electrically induced component
- \( \Theta \) thermally induced component

**Subscripts**
- \( k \) \( k \)th layer