DISTRIBUTED STRUCTURAL DYNAMICS CONTROL
OF FLEXIBLE MANIPULATORS—I.
STRUCTURAL DYNAMICS AND DISTRIBUTED
VISCOELASTIC ACTUATOR

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Abstract—Reducing structural dead weight has become of increasing importance in the design of new
generation lightweight and high-speed robot manipulators. However, due to the nature of structural
flexibility, the dynamic oscillation associated with robot structures can affect the operation accuracy and
precision. This work, in two parts, presents a study on the vibration control of elastic or flexible robot
structures. Effects of distributed passive (Part I) and active (in Part II) actuators on elastic robot structures
are studied. The proposed distributed passive viscoelastic actuator (in Part I) is a layer (or layers) of
viscoelastic polymer directly attached to the flexible robot element, the oscillation of which is to be
controlled. The passive actuator is activated by the oscillation of the robot structure and it automatically
dissipates vibration energy and constrains the undesirable motion to eliminate the disturbance and to
maintain a precise robot trajectory. A finite element program capable of analyzing flexible links is
developed. Results obtained from the finite element simulation are presented.

INTRODUCTION

Machines and robots nowadays need to be stronger, lightweight and capable of a very high speed of
operation. The high speeds have intensified the forces and torques acting on component parts of the
machines and robot structures, which result in higher amplitude oscillation of flexible components during
operation. This requires the structures to be strengthened beyond the basic loading requirements so that
they can withstand the higher forces and amplitudes, or it requires the introduction of additional damping
[1, 2]. Usually, implementing damping is the preferred technique as it does not increase weight and does not
cost as much.

There are two fundamental techniques of introducing additional damping: (1) the exciting force does
work in a damper (passive/viscous damper, passive actuator, dynamic absorber); and (2) the exciting
force or torque is opposed by another force produced by an auxiliary control device (actuator) [1, 2]. Part I
of this paper presents the effectiveness of two kinds of dampers/actuators: (1) a linear damper, a conven-
tional viscous damper associated with the dynamic structure itself, and (2) a nonlinear ‘passive’ visco-
elastic rubber actuator/damper, a motion-activated passive device. Viscoelastic rubber actuator/damper
includes a large range of materials with rubber-like properties. Examples include natural rubber,
neoprene and silicones. All have advantages for the vibration control specialist in that the stiffness of
the isolation component can easily be changed by an alteration in the hardness of the rubber. A large
number of applications can be found in the literature [3–7]. This viscoelastic actuator will be distributedly
applied on a flexible manipulator arm and its effect will be studied in Part I of this work—the variation
of passive damping caused by changes in the stiffness of the rubber material. The nonlinear ‘active’
damper/actuator will be discussed and evaluated in Part II of this work, which deals with integrated
distributed sensing and active control.

In this paper, a finite element (FE) program, the Nonlinear Dynamic Analysis Finite Element Program
(NDAFEP), is developed to simulate the dynamic response of flexible rotating robot manipulators.
In addition, a linear viscous damper and a ‘passive’ viscoelastic rubber damper/actuator are
implemented in the FE program. The first is assumed to be a proportional damping using the Rayleigh
model and the second is implemented as a ‘passive’ rubber damping element, modeled by a standard
linear model [2, 3, 6]. In the finite element analysis, an equivalent damping force derived from the damper/
actuator element is transferred to the right side of the system equations as a pseudo-force component
and is directly added to a linear excitation force vector in the numerical time-domain integration. The
dynamic response of the system is determined by a direct integration algorithm—the Wilson θ method
with minor modifications to accommodate the pseudo-forces derived from the damper/actuator
element [8].

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FINITE ELEMENT MODELING OF THE FLEXIBLE MANIPULATOR

In general, a geometrically nonlinear beam theory is required to account for the influence of centrifugal force on the bending stiffness in a rotating flexible manipulator or structure [9]. The higher order strain measures for a two-dimensional flexible rotating beam can be calculated by a successive approximation method using the linear beam theory [10]. With the stress-strain relations, Hamilton's principle can be employed to derive the equations of motion. The equations of motion may also be obtained from a suitable beam theory in which the absolute acceleration of a beam segment due to the rigid body rotation and the elastic deformation are both taken into consideration [11].

Consider a single-link flexible manipulator (modeled by a beam element) translating and rotating on the $X-Y$ plane. The $X-Y-Z$ coordinate system with its origin denoted by $O$ serves as the global coordinate system. The local coordinate system $x-y-z$ is defined by fixing its origin $O'$ to one end of the beam and by attaching the $x$-axis to the neutral axis of the beam (Fig. 1).

Note that the shear deformation and the rotary inerti are also included when deriving the equation of motion. That is, Timoshenko beam theory is used for the study of the elastic deformation. A free-body diagram of a differential beam element is shown in Fig. 2.

There are two elastic equations for the beam [12]

$$V_i = KAG\left\{\theta_i - \frac{\partial u_i}{\partial x}\right\}, \quad (1)$$

$$M_i = EI \frac{\partial \theta_i}{\partial x}, \quad (2)$$

where $V_i$ and $M_i$ are the shear force and the moment, respectively; $K$ is the Timoshenko shear coefficient; $E$ is the Young's modulus; $G$ is the shear modulus; $A$ is the cross-section area; $I$ is the moment of inertia; $\theta_i$ is the slope due to bending; and $\partial u_i/\partial x$ is the slope of the neutral axis. In addition to these two elastic equations, there are three dynamic equations for a differential beam element

$$F_x = a_x \cdot \rho A \, dx \quad (3a)$$

$$F_y = a_y \cdot \rho A \, dx \quad (3b)$$

$$M_i = \rho I (x + \theta_i) \cdot dx, \quad (3c)$$

where $F_x$ and $F_y$ are the two components of the resultant force corresponding to the $x$ and $y$ directions, respectively; $a_x$ and $a_y$ are, respectively, the accelerations of the beam segment in the $x$ and $y$ directions; $\rho$ is the mass of the mass density; and $x$ is the angular acceleration of the rigid body rotation of the beam. From the free-body diagram of the beam segment (Fig. 2), eqn (3) can be expressed as follows:

$$\begin{align*}
AE \frac{\partial u_i}{\partial x} + P_x &= \rho A a_x, \\
- \frac{\partial V_i}{\partial x} + P_y &= \rho A a_y, \\
\frac{\partial M_i}{\partial x} - V &= \rho I (x + \theta_i). \quad (4c)
\end{align*}$$

The linear acceleration of the segment $\ddot{a}$ consisting of two components, $\ddot{a}_r$ and $\ddot{a}_e$, is due to the rigid body motion and the elastic deformation. From the kinematics of the segment, we find that the acceleration of the beam can be calculated by the equation [11]

$$\ddot{a} = \ddot{a}_r + \ddot{a}_e \times (\dot{\omega} \times \dot{x}) + \ddot{\omega} \times \dot{k} + 2 \cdot \dot{\omega} \times \ddot{\omega} + \dddot{\omega}_e,$$  \quad (5)$$

where $\omega = \omega \cdot \dot{k}$ is the angular velocity of the rigid body rotation, $\dot{\omega} = \omega \cdot \dot{k}$ is the angular acceleration of the rigid body rotation; $\ddot{a}_r = a_x, \quad \dot{\ddot{a}}_r = \dot{a}_x, \quad \dddot{a} = \dot{a}_x$ is the linear acceleration of $a_x$; $\ddot{\omega}_e = u_x \cdot \dot{\omega} + u_y \cdot \dot{\omega}$ is the relative velocity of the beam segment with respect to the $x-y-z$ coordinate system; $\dddot{\omega}_e = \dot{u}_x \cdot \dot{\omega} + \dot{u}_y \cdot \dot{\omega}$ is the relative acceleration of the beam segment with respect to the $x-y-z$ coordinate system; and $\dot{x}$ is the position vector of the beam segment.

Examining the right-hand-side of eqn (5), the first term corresponds to the linear acceleration of the beam, the second and the third terms are associated with the rotation of the beam, the last term comes from the effect of elastic deformation, and the fourth
term is due to both the rigid body rotation and the
elastic deformation. Dividing eqn (5) into x and y
components gives
\[ a_x = a_{x'}, - \omega^2 u_x - a u_y - 2 u \omega_i + \bar{u}_x - \omega \omega^2 \]  
\[ a_y = a_{y'}, + \omega u_x + 2 u \omega_i + \bar{u}_y + \omega \omega^2 \]  
(6a)
(6b)
Substituting eqn (6) into eqn (4) yields the equation
of motion of the rotating link
\[ AE \frac{\partial^2 u_x}{\partial x^2} + P_x = \rho A (a_{x'} - \omega^2 u_x - a u_y) - 2 u \omega_i + \bar{u}_x - \omega \omega^2 \]  
\[ - \frac{\partial V}{\partial x} + P_x = \rho A (a_{y'} + \omega u_x + \omega^2 u_y + 2 u \omega_i + \bar{u}_y + \omega \omega^2) \]  
(7a)
(7b)
\[ \frac{\partial M}{\partial x} - V = \rho I (x + \bar{u}_x). \]  
(7c)
Next, in order to discretize the equations of
motion, the Galerkin method is used to derive
elemental matrices. The beam element start from
\[ x = x_1 \]  and end at \[ x = x_2 \]; then we have
\[ \int_{x_1}^{x_2} \left[ - AE \frac{\partial^2 u_x}{\partial x^2} - P_x + \rho A (a_{x'} - \omega^2 u_x - a u_y) - 2 u \omega_i + \bar{u}_x - \omega \omega^2 \right] \delta u_x \]  
\[ + \left[ - \frac{\partial V}{\partial x} + P_x + \rho A (a_{y'} + \omega u_x + \omega^2 u_y + 2 u \omega_i + \bar{u}_y + \omega \omega^2) \right] \delta u_x \]  
\[ + \left[ - \frac{\partial M}{\partial x} + V + \rho I (x + \bar{u}_x) \right] \delta \theta_x \right] \delta u_x = 0. \]  
(8)
The linear shape functions for the beam element are
\[ N_1(x) = (x - x_i)/L_x \]  
\[ N_2(x) = (x - x_i)/L_x \]  
(9a)
(9b)
where \( L_x = x_2 - x_1 \) is the length of the element.
The displacements within the beam element can be
expressed in terms of the nodal degrees of freedom by
the shape functions, i.e.
\[ u_x = N_1(x) U_{x1} + N_2(x) U_{x2} \]  
\[ u_y = N_1(x) U_{y1} + N_2(x) U_{y2} \]  
\[ \theta_x = N_1(x) \theta_{x1} + N_2(x) \theta_{x2}. \]  
(10a)
(10b)
(10c)
The arbitrary variations, \( \delta u_x, \delta u_y, \) and \( \delta \theta_x, \) can be
obtained from eqns (10)
\[ \delta u_x = N_1(x) \delta U_{x1} + N_2(x) \delta U_{x2} \]  
\[ \delta u_y = N_1(x) \delta U_{y1} + N_2(x) \delta U_{y2} \]  
\[ \delta \theta_x = N_1(x) \delta \theta_{x1} + N_2(x) \delta \theta_{x2}. \]  
(11a)
(11b)
(11c)
Substituting eqns (1), (2), (6), (9), (10), and (11) into
eqn (8) gives the equation of motion for the beam
element
\[ [M]_e \{ \ddot{U} \}_e + [C]_e \{ \dot{U} \}_e + [K]_e \{ U \}_e = \{ F \}_e, \]  
(12)
where the subscript e denotes element and g denotes
coriolis effect. \( [M]_e \) is the conventional mass
matrix. \( [C]_e \) is the skew-symmetric coriolis matrix, of which
the non-zero entries are functions of the mass of the
element and the rigid body angular velocity. \( [K]_e \) is
the stiffness matrix which is composed of the
conventional stiffness matrix and the dynamic geometric
stiffness matrix. The entries of the dynamic geometric
stiffness matrix are determined by the mass of the
element, the rigid body angular velocity and the
acceleration. \( \{ F \}_e \) is the applied force vector which consists
of the inertia forces associated with the rotation. \( \{ U \}_e \) is the
displacement vector.
The system matrices, \( [M], [C]_e, \) and \( [K] \), and
the force vector \( \{ F \} \) can be assembled using the element
matrices and vectors with appropriate transformation.
Again, the subscript g denotes coriolis effect.
The process is standard and not discussed here.
Finally, we obtain the equation of motion governing
the whole rotating flexible beam for finite element
analysis. The equation of motion is
\[ [M] \{ \ddot{U} \} + [C]_e \{ \dot{U} \} + [K] \{ U \} = \{ F \}. \]  
(13)
Next, the modified Wilson \( \theta \) method can be
employed to carry out the time-domain integration
of the system equations [8]. Since both the geometric
configuration and the rigid body motion of the beam
are usually different for each time step, the mass
matrix, the gyroscopic matrix, the stiffness matrix,
and the applied force vector are all time dependent.
As a result, updating the system matrices and vector
at each time step is essential. In order to obtain a
reasonable result, at each time step, the obtained \( \{ \dot{U} \}, \)
\( \{ U \}, \) and \( \{ U \} \) should also be modified for the following
time step to avoid a possible accumulated error.
The accumulated error may also be minimized by
providing small time steps.

**DEFINITION OF DAMPERS/ACTUATORS**

In this study, the effectiveness of various
dampers/actuators is evaluated. Two kinds of
dampers/actuators are implemented in the finite
element program NDAFE2 and compared for
vibration control efficiency, i.e. (1) a linear viscous (proportional) damper and (2) a ‘passive’ viscoelastic rubber damper/actuator. Each damper/actuator will be discussed separately.

**Linear viscous damper**

The linear viscous damper is associated with the structure of the mechanical system itself. The linear damping matrix is assumed to be proportional to the stiffness and mass matrices by Rayleigh’s coefficients \( \alpha \) and \( \beta \). It can be written as

\[
[C] = \alpha \cdot [M] + \beta \cdot [K].
\]  

(14)

The damping ratio of the \( n \)-th mode of the system is defined as

\[
\zeta_n = \frac{\alpha + \beta \cdot \omega_n^2}{2 \cdot \omega_n},
\]  

(15)

where \( \omega_n \) is the natural frequency of the \( n \)-th mode. The desirable damping ratio \( \zeta_n \) for the system can be determined by adjusting \( \alpha \) and \( \beta \). Usually, \( \zeta_n \) is determined experimentally, \( \alpha \) is set to 0, and \( \beta \) is calculated accordingly.

**Distributed nonlinear viscoelastic rubber damper/actuator**

A distributed viscoelastic rubber layer (or layers) is coupled with the flexible element to control the oscillation of a flexible robot arm (Fig. 3). The viscoelastic rubber damper/actuator is a popular and widely used passive damper. Various rubber dampers with different dynamic characteristics and physical shapes and made of different chemical compounds are available for a wide range of applications to vibration control. In general, rubber damping is introduced into the system in a passive way, i.e. a motion-activated passive controller which alleviates mechanical vibration, provides extra damping and thus improves overall system dynamics. The rubber damper/actuator is modeled by the standard linear model [2, 3, 6, 8] (Fig. 3b) and implemented by using a ‘rubber’ damper/actuator element in the NDAFEP.

A standard linear model consists of three parameters, a primary spring \( K_r \), in parallel with a serial of a secondary spring \( K_s \), and a viscous damper \( C \), as illustrated in Fig. 3b. When the rubber damper is

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Fig. 3. A flexible element with distributed viscoelastic actuator (a) and the standard linear model (b).

Fig. 4. Time history of the tip’s axial displacement.
subjected to a sinusoidal excitation at a frequency of \( \omega \), a complex stiffness \( \frac{F}{U} \) is defined as [2, 6, 8]

\[
\frac{F}{U} = \left[ K_s + \frac{K_v}{K_s^2 + (\omega \cdot C_s)^2} \right] \cdot \left[ \frac{C_s \cdot K_s^2}{K_s^2 + (\omega \cdot C_s)^2} \right] + j \omega \cdot \left[ \frac{C_s \cdot K_s^2}{K_s^2 + (\omega \cdot C_s)^2} \right].
\]  

(16)

In the finite element simulation, an equivalent stiffness, \( K^* \), corresponds to the real part and an equivalent damping, \( C^* \), multiplied by \( \omega \), represents the imaginary part. Thus, the equivalent rubber damping force, \( F_r \), derived from the rubber damper can be approximated by

\[
\{F_r(t)\} = \left[ K_s + \frac{K_v}{K_s^2 + (\omega C_s)^2} \right] \cdot \{\delta U'\}
\]

\[+ \left[ \frac{C_s \cdot K_s^2}{K_s^2 + (\omega C_s)^2} \right] \cdot \{\delta U'\},
\]  

(17)

or simply

\[
\{F_r(t)\} = \{C^*(\delta U')\} \cdot \{\delta U'\}
\]

\[+ \{K^*(\delta U')\} \cdot \{\delta U'\},
\]  

(18)

where \( \{C^*(\delta U')\} \) and \( \{K^*(\delta U')\} \) are the equivalent rubber damping and stiffness matrices, respectively. \( \{\delta U'\} \) and \( \{\delta U'\} \) are the relative velocity and relative displacement vectors, which can be defined as

\[
\{\delta U'\} = \{U_i\} - \{U_j\},
\]

(19a)

\[\{\delta U'\} = \{U_i\} - \{U_j\},
\]  

(19b)

where \( i \) and \( j \) are the two adjacent nodes connected by the damper element. To generalize the equation, a set of step functions \( S(\psi) \) and \( S(\xi) \) can also be applied to eqn (18) as

\[
\{F_r(t)\} = \{C^*(\delta U')\} \cdot \{\delta U'\} \cdot S(\psi)
\]

\[+ \{K^*(\delta U')\} \cdot \{\delta U'\} \cdot S(\xi)
\]  

(20)

or

\[
\{F_r(t)\} = \left[ \frac{C_s \cdot K_s^2}{K_s^2 + (\omega C_s)^2} \right] \cdot \{\delta U'\}
\]

\[- \{\psi\} \cdot S(\psi) + \left[ \frac{K_s + \frac{K_v(\omega C_s)^2}{K_s^2 + (\omega C_s)^2}}{K_s^2 + (\omega C_s)^2} \right]
\]

\[\times \{\delta U'\} \cdot \{U_i\} - \{U_j\} \cdot S(\xi)
\]  

(21)

which implies that the damping term activates when \( \{\delta U'\} \geq \{\phi\} \) and the stiffness term activates when \( \{\delta U'\} \geq \{\xi\} \). However, in practice the damping term and the stiffness term usually start activating at the same time.

**EQUATION OF MOTION WITH NONLINEAR VISCOELASTIC ACTUATORS**

For a flexible structure or mechanical system with \( N \) degrees of freedom, the generalized nonlinear matrix equation of motion with linear viscous damping and distributed viscoelastic damping can be formulated as

\[
[M]\{\ddot{U}\} + [C]\{U\} + [K]\{U\} = \{F(t)\} + \{F_r(t)\}
\]  

(22)

Fig. 5. Time history of the tip's transverse displacement.
or

\[
[M]\{\ddot{U}\} + [C]\{\dot{U}\} + [K]\{U\} + [F](t) = 0
\]

\[
= \left[ \frac{C_{1}K_{0}}{K_{1}^{2} + (\omega C_{1})^{2}} \right]
\times \left\{ \{\ddot{U}_{1}\} - \{\ddot{U}_{1}\} - \{\dot{\theta}\} \cdot S\{\theta\} \right. \\
+ \left. \left[ K_{1} + \frac{K_{0}(\omega C_{1})}{K_{1}^{2} + (\omega C_{1})^{2}} \right] \times \left\{ \{\ddot{U}_{1}\} - \{\ddot{U}_{1}\} - \{\dot{\xi}\} \cdot S\{\xi\} \right\}. \right. \tag{23}
\]

In the case study, the dynamics of the flexible manipulator arm and these damping effects are evaluated separately.

**EXAMPLES**

In order to evaluate the effectiveness of the linear viscous damper and the nonlinear distributed 'passive' viscoelastic damper/actuator, a single-link flexible robot arm is used as an example. As discussed earlier, the hardness (\(K_{0}\)) and damping (\(C_{1}\)) of a viscoelastic damper determine the dynamic characteristics and effectiveness. The beam is made of steel with the properties listed in Table 1.

In the finite element analysis, the beam is divided into eight beam elements with the first node hinged at the origin. The flexible beam is input at an angular acceleration \(\ddot{\theta}(t)\) for a duration of 0.3 sec followed by a deceleration of the same period. The excitation profile is defined as follows.

<table>
<thead>
<tr>
<th>Table 1. Material and geometric properties of the flexible arm</th>
</tr>
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<tbody>
<tr>
<td>(E) (Young's modulus) \hspace{1em} 2.97 \times 10^{11} \text{ N/m}^2</td>
</tr>
<tr>
<td>(\rho) (density) \hspace{1em} 7825 \text{ kg/m}^3</td>
</tr>
<tr>
<td>(l) (length) \hspace{1em} 60.96 cm</td>
</tr>
<tr>
<td>(w) (width) \hspace{1em} 2.54 cm</td>
</tr>
<tr>
<td>(h) (thickness) \hspace{1em} 0.508 cm</td>
</tr>
<tr>
<td>(G) (shear modulus) \hspace{1em} 8.27 \times 10^{4} \text{ N/m}^2</td>
</tr>
</tbody>
</table>

(1) \((0 \leq t \leq 0.3\) sec): acceleration phase

\[
\ddot{\theta}(t) = (1/0.3)(t/2)^{2} + (0.3/2\pi)^{2} \times \cos(2\pi(0.3) - 1)] \text{ rad} \tag{24a}
\]

\[
\dot{\omega}(t) = (1/0.3)[r - (0.3/2\pi)] \times \sin(2\pi(0.03)] \text{ rad/sec} \tag{24b}
\]

\[
\alpha(t) = (1/0.3)[1 - \cos(2\pi(0.3)] \text{ rad/sec}^2 \tag{24c}
\]

(2) \((0.3 < t \leq 0.6\) sec): deceleration phase

\[
\ddot{\theta}(t) = 0.3 - (1/0.3)(0.6 - t)^{2}/2 + (0.3/2\pi)^{2} \times \cos(2\pi(0.6 - t)/0.3 - 1)] \text{ rad} \tag{25a}
\]

\[
\dot{\omega}(t) = (1/0.3)(0.6 - t) - (0.3/2\pi) \times \sin(2\pi(0.6 - t)/0.03\] \text{ rad/sec} \tag{25b}
\]

\[
\alpha(t) = -(1/0.3)(1 - \cos[2\pi(0.6 - t)/0.3]) \text{ rad/sec}^2 \tag{25c}
\]

**Fig. 6.** Time history of the tip’s rotation angle.
(3) \((t \geq 0.6 \text{ sec})\): stop

\[\theta(t) = 0.3 \text{ rad}\]  \hspace{1cm} (26a)

\[\omega(t) = 0.0 \text{ rad/sec}\]  \hspace{1cm} (26b)

\[a(t) = 0.0 \text{ rad/sec}^2\]  \hspace{1cm} (26c)

Three cases are studied here: Case 1 is a study on the dynamics of the flexible robot arm without damping; Case 2 includes the linear viscous damping; and Case 3 includes the nonlinear distributed passive viscoelastic dampers/actuators.

**Case 1: dynamics of an undamped flexible robot arm**

The flexible arm rotates according to the rotation profile defined by eqns (24)–(26). The time histories of the tip’s axial and transverse displacements as well as the rotation angle are shown in Figs 4–6. The displacements are calculated by the ‘large deformation’ approach, in which the axial deformation due to the bending and the transverse deformation due to the axial deformation are considered. Note that none of the damping effects was considered in this case. From these time histories, we find that the transverse displacement is of the order of approximately \(10^3\) times that of the axial displacement.

The first natural frequency of the transverse vibration can be calculated by

\[f = \frac{3.52}{2\pi \cdot L} \sqrt{\frac{EI}{\rho A}} = 11.35 \text{ Hz}.\]  \hspace{1cm} (27)

From the time history of the transverse displacement (Figs 4 and 6), we find that the steady state vibration frequency equals the first mode natural frequency. (The oscillation in Fig. 5 did not show because the scale is too small.) It is also noted that the vibration of the flexible arm maintains the same frequency and amplitude after the steady state is reached due to zero system damping. Figures 4 and 6 both show the antisymmetric transient responses in the acceleration and deceleration phases; however, the steady state responses show the same pattern. It is also observed that the flexible arm oscillates even during the deceleration phase at the first natural frequency.

**Case 2: a flexible arm with linear viscous damping**

It is well known that the linear viscous damping is associated with the structure itself. Thus, in this case, a 2.0\% damping was assigned and introduced into the flexible arm by assuming \(\alpha = 0.0\) and \(\beta = 5.61 \times 10^{-4}\). The proportional damping matrix \([C]\) was then determined using eqn (14). The transverse oscillation time history is shown in Fig. 7.

It is observed that the maximum transient response occurring at around 0.047 sec was reduced and the steady state oscillation was damped and finally converged to zero. The actual damping ratio of the system can be estimated by the logarithmic decrement method: \(\zeta = \ln(y_i/y_{i+1})/2\pi\), where \(\zeta\) is the damping ratio and \(y_i\) and \(y_{i+1}\) are the amplitudes at the \(i\)th and the \((i + 1)\)th peaks, respectively. From calculation, we found that the damping ratio of the system equals 2.0\%, which is identical to the value assigned. Note that the system’s damping ratio has been verified by the logarithmic decrement method. This method was used throughout the study to estimate the damping

![Fig. 7. Time history of a flexible arm with linear viscous damping.](image-url)
ratio of the system as different distributed dampings were incorporated with the flexible arm.

Case 3: A flexible arm with nonlinear distributed viscoelastic actuator

A layer (or layers) of viscoelastic rubber can be attached on the surface of the arm to control the oscillation of the flexible arm [2, 4, 13]. It is assumed that the viscoelastic damper acts in both tension and compression. There are three parameters in a viscoelastic damper/actuator model: the primary stiffness $K_1$, the secondary stiffness $K_2$, and the viscous damper $C$. In this study, the second stiffness $K_2$ is set to be three times the primary stiffness $K_1$ (rubber damper) [2, 3]. Thus, the three parameters are then reduced to two parameters: the primary stiffness and the damping ratio of the rubber damper. The rubber damping $C$ is determined by $C = 2\zeta\sqrt{K_1M}$, where $M$ is the applied concentrated nodal mass and $\zeta$ is the rubber damping ratio. Note that the rubber damping

![Figures](image1.png)

(a) $K_1 = 2$ lb/in.

![Figures](image2.png)

(b) $K_1 = 7$ lb/in.

Fig. 8. Time histories of a flexible arm with distributed viscoelastic damper/actuator.
ratio $\zeta$ is assumed to be 25% in this study. In order to obtain the effect of the primary stiffness, the damping ratios of the rubber dampers were set to be 25%, and only the primary stiffnesses of the dampers were changed. Figure 8a and b show the effectiveness of two different distributed viscoelastic dampers with (a) $K_1 = 2$ lb/in. and (b) $K_2 = 7$ lb/in. (It should be noted that when testing the viscoelastic damping effect the linear viscous damping was set to zero.)

The transient maximum displacement of the flexible arm was evaluated for different viscoelastic rubber dampers and they are plotted in Fig. 9. It is shown that the transient deflection is reduced when heavier rubber dampers are used. The steady-state response is also reduced (Fig. 8).

**SUMMARY AND CONCLUSIONS**

In this paper, the dynamics of a flexible robot manipulator were evaluated using a finite element method. The transient response and the steady-state oscillation were controlled by a distributed viscoelastic rubber damper. The geometric nonlinearity due to the large rotation was calculated at any time instant and the system matrices were reformulated in the time-domain integration. The nonlinearity associated with the distributed viscoelastic rubber damper modeled by a standard linear model was approximated by a pseudo-force method.

The dynamic characteristics of a flexible robot arm were evaluated using the developed finite element program. The steady-state axial elongation and the oscillation frequency were validated by theoretical calculations. The linear viscous damping was also verified using the logarithmic method. A distributed passive viscoelastic damper was proposed and its effectiveness was evaluated. It was shown that both transient and steady-state oscillations were reduced when using the viscoelastic rubber dampers. However, it should be noted that a very heavy damper could lose its control effect when exceeding an optimal value. This optimal design requires further investigation in the future.

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**REFERENCES**


