THEORETICAL ANALYSIS OF A MULTI-LAYERED THIN SHELL COUPLED WITH PIEZOELECTRIC SHELL ACTUATORS FOR DISTRIBUTED VIBRATION CONTROLS

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Structural dynamics and controls of distributed mechanical systems have drawn much attention in recent years. In this paper, a multi-layered thin shell coupled with an active distributed vibration actuator—polymeric piezoelectric polyvinylidene fluoride (PVDF)—is proposed and evaluated. Dynamic equations for the generalized multi-layered thin shell coupled with the polymeric piezoelectric actuator are derived based on Love's theory and Hamilton's principle. Each layer of the shell can be a polymeric piezoelectric control layer subjected to feedback voltages resulting in a local control force to suppress the vibration of the shell. To demonstrate the derived equations, a cantilever beam coupled with the polymeric piezoelectric actuator is derived by directly simplifying the modified Love's equations. An experimental model was also built and tested in laboratory to validate the results.

1. INTRODUCTION

Due to the rapid development in aerospace exploration, structural dynamics and controls of distributed mechanical systems have drawn much attention in recent years. Because of lack of structural and environmental damping in space, fast motion or high-speed maneuver of a flexible space structure often leads to an uncontrolled vibration of large amplitude which is highly destructive. In addition, the prolonged vibration in the flexible space structure can introduce an excessive material fatigue, affect the operational accuracy, and hence directly reduce the life cycle. Thus, it is highly desirable to control the excessive vibration and to effectively stabilize the space system after any maneuver. One method is to stiffen the structures enough to withstand these vibrations, but this leads to strength and weight requirements far in excess of those demanded by the initial payload. As a result, an effective vibration control is crucial under such circumstances. Means of controlling the flexible space structure are very much in demand today. There are, in general, two categories of techniques available to the vibration control specialist: (1) the "passive" control, e.g., metal springs, viscoelastic damper, dynamic absorber, auxiliary mass damper, etc. [1-3]; and (2) the "active" control in which the effects of the undesirable force are counteracted by an auxiliary mechanism, e.g., electromechanical, pneumatic, and electromagnetic actuators, etc. [4, 5]. In this study, an electromechanical polymer—piezoelectric polyvinylidene-fluoride (PVDF)—has been used as an active distributed vibration control device to suppress the vibration of distributed mechanical systems.

In general, a piezoelectric material responds to mechanical force and/or pressure and generates an electrical charge, which is called the direct piezoelectric effect [6], e.g., accelerometer, pressure transducer, etc. Conversely, application of an electric field to the material can produce mechanical stress or strain which is referred to the reciprocal, or converse piezoelectric effect, e.g., vibration suppressor. Polyvinylidene fluoride (PVDF)
is a tough and pliant piezoelectric polymer. It is flexible and can be produced in large sheets and complex geometries. In contrast to this, conventional piezoelectrics like ceramics and naturally occurring crystals are dense, brittle and difficult to fabricate into complex shapes. In addition to this, PVDF has a dynamic sensitivity that runs from about $10^{-4}$ to $10^{4}$ N/m, (a 286 dB range) and its maximum response frequency is in the GHz range [7, 8]. All these qualities, along with low production cost have made PVDF very attractive for sensing and active vibration control of distributed mechanical and structural systems.

Applications of the PVDF polymer to dynamic measurements were studied by Tzou and Pandita [9], Miller and Hubbard [10], and many other researchers in the past decade [8, 11, 12]. Distributed sensing phenomena of a PVDF shell were recently investigated theoretically [13] and numerically [14]. A PVDF polymer was also used as a distributed actuator for a Bernoulli-Euler beam [15]. Sirlin studied the vibration isolation of a spacecraft using the piezoelectric polymer [16]. Tzou and Gadre evaluated a uniaxial active vibration isolation using the polymeric PVDF as an active isolator with velocity feedbacks [17]. Gosebruch also used a piezoceramic driver to actively enhance the damping in grinding machines [18]. Tzou and Tseng studied the active distributed vibration control by a finite element method [19] and distributed sensing and active vibration control of a thin shell [20]. The above studies were concentrated on relatively simple geometries. Generalized theoretical development on multi-layered shells needs to be further explored.

This paper presents a theoretical development of a multi-layered thin shell coupled with a piezoelectric actuator for active distributed vibration suppression. Starting from some generalized assumptions, equations for a generalized multi-layered thin shell coupled with the polymeric piezoelectrets are derived according to Love's theory [21] and Hamilton's principle [22]. Each layer of the shell can be a polymeric piezoelectric layer subjected to feedback voltages resulting in local control forces (due to the converse piezoelectric effect) to suppress the oscillation of the shell. The derived shell equations can be simplified to account for different geometries, such as shallow shell, cylinder, plate, sphere, beam, etc. [20, 22, 23]. To demonstrate the derived equations, a cantilever beam coupled with the distributed polymeric piezoelectric actuator is used as an example. System dynamic equation of the beam will be derived by directly simplifying the modified Love equations. An experimental model also has been built and tested in the laboratory to validate the results. Effectiveness of the active distributed piezoelectric vibration controller is evaluated.

2. ACTIVE VIBRATION CONTROL OF A MULTI-LAYERED SHELL

First, the necessary theoretical background will be developed. Love's equations of motion [21] for thin shells will be modified to account for multiple layers, some of which may be subjected to localized electromechanical control forcing functions in the form of an excitation voltage applied to the piezoelectric layers. Figure 1 shows a schematic of the multi-layered shell with its curvilinear co-ordinate system. For the most part, only the modifications are shown here. For detailed derivation the reader should refer to the references [22–24]. The nomenclature is defined in the Appendix.

2.1. ASSUMPTIONS FOR THE MULTI-LAYERED THIN SHELL

A number of assumptions are made for the derivation of system equations, as follows.

1. Several thin, isotropic shell layers one above the other are bonded together: even though the shell is made up of several layers, it can still be considered a thin shell. For a biaxially oriented PVDF polymer subjected to small deformation, it is assumed isotropic.
2. The neutral surfaces of the layers are all parallel to each other and the layers are all of the same dimensions in $\alpha_1$ and $\alpha_2$. However, the thicknesses can be different. This automatically means that the Lamé parameters for all the layers are the same.

3. All transverse distances and displacements are measured from a reference surface. This surface will usually be the "composite" neutral surface.

4. In the absence of any localized forces, the layers together move as a single layer thin shell.

5. If a forcing function (for example, a sinusoidally varying voltage applied to a piezoelectric sheet) is applied locally to one (or several) layer, then the (added) induced deformations and strains appear in that layer only. (However, the extra forces and moments will affect the rest of the structure.)

6. The composite thin shell is composed of several thin layers and the radii of curvature vary negligibly over the different layers.

7. Only the in-plane forces and moments are considered [22]. The resultant normal forces $N_{jk}$ and moments $M_{jk}$ for the entire layered shell will be evaluated in the following way, for $j, k = 1, 2$:

$$N_{jk} = \int_{\delta_i}^{\delta_j} \sigma_{jk} \, d\alpha_3 = \sum_{i=1}^{n} \int_{\delta_{i}}^{\delta_{i+1}} \sigma_{jk} \, d\alpha_3 = \sum_{i=1}^{n} N_{jk},$$

$$M_{jk} = \int_{\delta_i}^{\delta_j} \alpha_3 \sigma_{jk} \, d\alpha_3 = \sum_{i=1}^{n} \int_{\delta_{i}}^{\delta_{i+1}} \alpha_3 \sigma_{jk} \, d\alpha_3 = \sum_{i=1}^{n} M_{jk}.$$  

Love's equations are developed for each layer. Then, to obtain Love's equations for the entire layered shell, the corresponding equations are summed over all layers to obtain the equations in terms of the resultant normal forces and moments.

2.2. ANALYSIS FOR AN ARBITRARY $i$TH LAYER

Suppose there is a localized forcing function at the $i$th layer (in the form of a voltage in the case of piezoelectricity). Then, the induced (localized) strains for that layer are denoted by $\epsilon_{i1}$ and $\epsilon_{i2}$. The induced (localized) displacements for that layer are denoted by $\Delta_{1i}, \Delta_{2i},$ and $\Delta_{3i}$. In this section, basic equations for a generalized shell in a curvilinear co-ordinate system ($\alpha_1, \alpha_2, \alpha_3$) will be defined first. Then, modifications to account for the piezoelectric control force in the $i$th layer are to be derived.

The geometry of a shell element is defined by its fundamental form [22-24]

$$ds^2 = A_{1i}^2 (d\alpha_1)^2 + A_{2i}^2 (d\alpha_2)^2,$$

where $ds$ is a infinitesimal distance and $A_1$ and $A_2$ are the Lamé parameters. The basic stress-$\sigma$-strain-$\epsilon$ relationships are defined as

$$\epsilon_{11} = (1/Y_1)[\sigma_{11} - \mu_i(\sigma_{22} + \sigma_{33})], \quad \epsilon_{22} = (1/Y_1)[\sigma_{22} - \mu_i(\sigma_{11} + \sigma_{33})],$$

(4a, b)
where \( \sigma_y \) is the stress, \( \mu \) is the Poisson ratio, and \( Y \) is the Young's modulus. According to Love's assumptions [21] that the normal stress in \( \alpha_3 \) direction of a thin shell is negligible, i.e., \( \sigma_{33} = 0 \), one can define

\[
g_{ij}(\alpha_1, \alpha_2, \alpha_3) = A_j^2(1 + \alpha_3/R_j)^2, \quad j = 1, 2, \quad g_{33}(\alpha_1, \alpha_2, \alpha_3) = 1, \quad (5a, b)
\]

where \( \alpha_3 \) is the distance measured from the neutral surface, \( R_j \) and is the curvature in the \( \alpha_j \) direction. When a localized forcing function is applied to the \( j \)th layer, the induced displacements and strains appear in that layer only. Thus, according to thin-shell theory and Love's simplifications, the resultant displacement \( U_n \) can be expressed as

\[
U_i, (\alpha_1, \alpha_2, \alpha_3) = u_i(\alpha_1, \alpha_2) + \alpha_3 \beta_i^*(\alpha_1, \alpha_2) + \Delta_i(\alpha_1, \alpha_2), \quad (6a)
\]

\[
U_j, (\alpha_1, \alpha_2, \alpha_3) = u_j(\alpha_1, \alpha_2) + \alpha_3 \beta_j^*(\alpha_1, \alpha_2) + \Delta_j(\alpha_1, \alpha_2), \quad (6b)
\]

where superscript \( \tau \) denotes the (total) resultant effect and \( \beta_i^* \) is the total angle of rotation of the normal to the neutral surface when deformed. Note the use of \( \beta_i^* \) and \( \beta_j^* \) to indicate that the effects of the induced strains and displacements have been included. These two variables need to be determined by using Love's assumptions. The resultant displacement in the \( \alpha_3 \) direction is

\[
U_3, (\alpha_1, \alpha_2, \alpha_3) = u_3(\alpha_1, \alpha_2) + \Delta_3(\alpha_1, \alpha_2, \alpha_3). \quad (6c)
\]

For a thin shell, the normal strains are negligible: i.e.,

\[
\varepsilon_{13} = 0, \quad \varepsilon_{23} = 0. \quad (7)
\]

Thus

\[
\varepsilon_{13} = A_1 \left( 1 + \frac{\alpha_3}{R_1} \right) \frac{\partial}{\partial \alpha_3} \left[ \frac{U_{11}}{A_1(1 + \alpha_3/R_1)} \right] + \frac{1}{A_1(1 + \alpha_3/R_1)} \frac{\partial U_3}{\partial \alpha_1} = 0. \quad (8)
\]

Substituting the modified Love simplifications, equations (7), and retaining the effects, equations (5a) and (6c), of the induced localized forcing function with \( \varepsilon_{13} = 0 \), give

\[
\sqrt{g_{11}} \frac{\partial}{\partial \alpha_3} \left[ u_1 + \Delta_1 + \alpha_3 \beta_1^{*11} \right] + \frac{1}{\sqrt{g_{11}}} \frac{\partial}{\partial \alpha_1} (u_3 + \Delta_3) = 0
\]

\[
\Rightarrow \beta_i^* = \frac{u_1}{R_1} \frac{1}{A_1} \frac{\partial u_3}{\partial \alpha_1} + \frac{\Delta_1}{R_1} \frac{1}{A_1} \frac{\partial \Delta_3}{\partial \alpha_1}. \quad (9, 10)
\]

Note that the term in parentheses appears due to an unequal loading (electromechanical control force) on the layer. For example, this could occur if a spatially varying voltage is applied to a piezoelectric layer. The total rotation angle can be further divided into two components: (1) the original \( \beta_i \), and (2) the induced \( \beta_i^* \) terms (the superscript \( \tau \) denotes the induced effect).

\[
\beta_i^* = \beta_i + \beta_i^* \quad (11)
\]

where

\[
\beta_i = \frac{u_i}{R_1} \frac{1}{A_1} \frac{\partial u_i}{\partial \alpha_1}, \quad \beta_i^* = \frac{\Delta_i}{R_1} \frac{1}{A_1} \frac{\partial \Delta_i}{\partial \alpha_1}. \quad (12)
\]

Similarly,

\[
\beta_i^* = \left( \frac{u_i}{R_2} \frac{1}{A_2} \frac{\partial u_i}{\partial \alpha_2} \right) + \left( \frac{\Delta_i}{R_2} \frac{1}{A_2} \frac{\partial \Delta_i}{\partial \alpha_2} \right) = \beta_i + \beta_i^*. \quad (13)
\]
Now one can define the total rotation angle in terms of the original and the induced components. Next, one needs to evaluate the other strains in the same way:

$$
\varepsilon_{11}^{\ast} = \frac{1}{\sqrt{E_{11}}} \left[ \frac{\delta U_{11}}{\delta \alpha_{1}} + \frac{U_{21}}{A_{2}} \frac{\partial A_{1}}{\partial \alpha_{2}} + U_{31} \frac{A_{1}}{R_{1}} \right]. \tag{14}
$$

Substituting the modified Love simplifications from equations (6a, b, c) into equation (14) and assuming that the distance $\alpha_{3}$ is negligible compared with the radii of curvature, i.e.,

$$
\alpha_{3}/R_{1} \ll 1, \quad \alpha_{3}/R_{2} \ll 1, \tag{15}
$$

yields two in-plane strains, $\varepsilon_{11}^{\ast}$, and $\varepsilon_{22}^{\ast}$:

$$
\varepsilon_{11}^{\ast} = \frac{1}{A_{1}} \frac{\delta}{\delta \alpha_{1}} \left( u_{1} + \Delta_{1} + \alpha_{3} \beta_{1}^{0} \right) + \frac{1}{A_{1} A_{2}} \left( u_{2} + \Delta_{2} + \alpha_{3} \beta_{2}^{0} \right) + \frac{u_{3} + \Delta_{3}}{R_{1}}, \tag{16}
$$

$$
\varepsilon_{22}^{\ast} = \frac{1}{A_{2}} \frac{\delta}{\delta \alpha_{2}} \left( u_{2} + \Delta_{2} + \alpha_{3} \beta_{2}^{0} \right) + \frac{1}{A_{1} A_{2}} \left( u_{1} + \Delta_{1} + \alpha_{3} \beta_{1}^{0} \right) + \frac{u_{3} + \Delta_{3}}{R_{2}}, \tag{17}
$$

$\varepsilon_{33}$ is defined by

$$
\varepsilon_{33}^{\ast} = \frac{\partial U_{3}}{\partial \alpha_{3}} = \frac{\partial \Delta_{3}}{\partial \alpha_{3}}. \tag{18}
$$

In the case of a piezoelectric layer, this would be the strain in the direction of a transversely applied electric field.

### 2.2.1. Membrane and bending strains

In general, total strain is induced by two components, (1) in-plane strains $\varepsilon_{m}^{\ast}$, and (2) bending strains $k_{m}^{\ast}$. The membrane and bending strains in the $\alpha_{1}$ and $\alpha_{2}$ directions can be separated:

$$
\varepsilon_{11}^{\ast} = \frac{1}{A_{1}} \frac{\delta}{\delta \alpha_{1}} \left( u_{1} + \Delta_{1} \right) + \frac{u_{2} + \Delta_{2}}{A_{1} A_{2}} \frac{\partial A_{1}}{\partial \alpha_{2}} + \frac{u_{3} + \Delta_{3}}{R_{1}}, \tag{19}
$$

$$
\varepsilon_{22}^{\ast} = \frac{1}{A_{2}} \frac{\delta}{\delta \alpha_{2}} \left( u_{2} + \Delta_{2} \right) + \frac{u_{1} + \Delta_{1}}{A_{1} A_{2}} \frac{\partial A_{1}}{\partial \alpha_{1}} + \frac{u_{3} + \Delta_{3}}{R_{2}}, \tag{20}
$$

$$
\varepsilon_{12}^{\ast} = \frac{A_{2}}{A_{1}} \frac{\partial}{\partial \alpha_{1}} \left( \frac{u_{2} + \Delta_{2}}{A_{2}} \right) + \frac{A_{1}}{A_{2}} \frac{\partial}{\partial \alpha_{2}} \left( \frac{u_{1} + \Delta_{1}}{A_{1}} \right); \tag{21}
$$

$$
\kappa_{11}^{\ast} = \frac{1}{A_{1}} \frac{\delta}{\delta \alpha_{1}} \frac{\beta_{1}^{0}}{A_{1}}, \quad \kappa_{22}^{\ast} = \frac{1}{A_{2}} \frac{\delta}{\delta \alpha_{2}} \frac{\beta_{2}^{0}}{A_{2}} + \frac{A_{1}}{A_{2}} \frac{\partial}{\partial \alpha_{1}} \left( \frac{\beta_{1}^{0}}{A_{1}} \right), \quad \kappa_{12}^{\ast} = \frac{A_{2}}{A_{1}} \frac{\delta}{\delta \alpha_{1}} \left( \frac{\beta_{2}^{0}}{A_{2}} \right) + \frac{A_{1}}{A_{2}} \frac{\partial}{\partial \alpha_{2}} \left( \frac{\beta_{1}^{0}}{A_{1}} \right); \tag{22, 23}
$$

One can now separate and rewrite the induced membrane strains $\varepsilon_{m}^{\ast}$, and bending strains $k_{m}^{\ast}$ due to the localized forcing function on the $i$th layer. (Note that the induced terms have a superscript "*".)

$$
\varepsilon_{11}^{i} = \frac{1}{A_{1}} \frac{\delta}{\delta \alpha_{1}} \left( \frac{\Delta_{1}}{A_{1}} \right) + \frac{\Delta_{2}}{A_{2}} + \frac{\partial A_{1}}{\partial \alpha_{2}} + \frac{\Delta_{3}}{R_{1}}, \quad \varepsilon_{22}^{i} = \frac{1}{A_{2}} \frac{\delta}{\delta \alpha_{2}} \left( \frac{\Delta_{2}}{A_{2}} \right) + \frac{\Delta_{1}}{A_{1}} + \frac{\partial A_{2}}{\partial \alpha_{1}} + \frac{\Delta_{3}}{R_{2}}, \tag{25, 26}
$$

$$
\varepsilon_{12}^{i} = \frac{A_{2}}{A_{1}} \frac{\delta}{\delta \alpha_{1}} \left( \frac{\Delta_{2}}{A_{2}} \right) + \frac{A_{1}}{A_{2}} \frac{\partial}{\partial \alpha_{2}} \left( \frac{\Delta_{1}}{A_{1}} \right); \tag{27}
$$
$$k'_{1i} = \frac{1}{A_1} \frac{\partial^2 \beta_i}{\partial \alpha^2} + \frac{\beta_i}{A_1 A_2} \frac{\partial A_1}{\partial \alpha}$$

$$= \frac{1}{A_1} \frac{\partial}{\partial \alpha} \left[ \frac{A_i}{R_i} \frac{1}{A_1} \frac{\partial \Delta_i}{\partial \alpha} \right] + \frac{1}{A_1 A_2} \frac{\partial}{\partial \alpha} \left[ \frac{A_i}{R_i} \frac{1}{A_2} \frac{\partial \Delta_i}{\partial \alpha} \right] \frac{\partial A_1}{\partial \alpha}$$

$$k'_{22} = \frac{1}{A_2} \frac{\partial^2 \beta_i}{\partial \alpha^2} + \frac{\beta_i}{A_1 A_2} \frac{\partial A_2}{\partial \alpha}$$

$$= \frac{1}{A_2} \frac{\partial}{\partial \alpha} \left[ \frac{A_i}{R_i} \frac{1}{A_2} \frac{\partial \Delta_i}{\partial \alpha} \right] + \frac{1}{A_1 A_2} \frac{\partial}{\partial \alpha} \left[ \frac{A_i}{R_i} \frac{1}{A_1} \frac{\partial \Delta_i}{\partial \alpha} \right] \frac{\partial A_2}{\partial \alpha}$$

$$k'_{12} = \frac{A_i}{A_1} \frac{\partial}{\partial \alpha} \left( \frac{\beta_i}{A_2} \right) + \frac{A_i}{A_2} \frac{\partial}{\partial \alpha} \left( \frac{\beta_i}{A_1} \right)$$

$$= \frac{A_i}{A_1} \frac{\partial}{\partial \alpha} \left[ \frac{1}{A_2} \left( \frac{A_i}{R_i} \frac{1}{A_2} \frac{\partial \Delta_i}{\partial \alpha} \right) \right] + \frac{A_i}{A_2} \frac{\partial}{\partial \alpha} \left[ \frac{1}{A_1} \left( \frac{A_i}{R_i} \frac{1}{A_1} \frac{\partial \Delta_i}{\partial \alpha} \right) \right]$$

If the extra forcing function on the ith layer does not vary spatially, then the k' terms will vanish. Now the total strains can be represented as a summation of the original strain and the piezoelectric induced strain:

$$\varepsilon'_{11} = \varepsilon_{11}^0 + \varepsilon_{11}^\phi, \quad \varepsilon'_{22} = \varepsilon_{22}^0 + \varepsilon_{22}^\phi, \quad \varepsilon'_{12} = \varepsilon_{12}^0 + \varepsilon_{12}^\phi$$

$$k'_{11} = k_{11} + k_{11}^\phi, \quad k'_{22} = k_{22} + k_{22}^\phi, \quad k'_{12} = k_{12} + k_{12}^\phi$$

Thus,

$$\varepsilon'_{11} = \varepsilon_{11}^0 + \alpha \varepsilon_{11}^\phi + \alpha \varepsilon_{11}^\phi, \quad \varepsilon'_{22} = \varepsilon_{22}^0 + \alpha \varepsilon_{22}^\phi + \alpha \varepsilon_{22}^\phi, \quad \varepsilon'_{12} = \varepsilon_{12}^0 + \alpha \varepsilon_{12}^\phi + \alpha \varepsilon_{12}^\phi$$

2.2.2. Stress-strain relationships

The stresses can now be analyzed in terms of strains:

$$\sigma'_{11} = \frac{Y_i}{1 - \mu_i^2} \left[ \varepsilon'_{11} + \mu \varepsilon'_{22} + \mu \varepsilon_{33} \right]$$

$$= \frac{Y_i}{1 - \mu_i^2} \left( \varepsilon_{11}^0 + \mu \varepsilon_{22}^0 + \mu \varepsilon_{33} \right) + \frac{Y_i}{1 - \mu_i^2} \left( \varepsilon_{11}^\phi + \mu \varepsilon_{22}^\phi + \mu \varepsilon_{33}^\phi \right) = \sigma_{11} + \sigma_{11}^\phi$$

where

$$\sigma_{11} = \frac{Y_i}{1 - \mu_i^2} \left( \varepsilon_{11}^0 + \mu \varepsilon_{22}^0 \right), \quad \sigma_{11}^\phi = \frac{Y_i}{1 - \mu_i^2} \left( \varepsilon_{11}^\phi + \mu \varepsilon_{22}^\phi \right)$$(36a, b)

Similarly,

$$\sigma'_{22} = \sigma_{22} + \sigma_{22}^\phi, \quad \sigma'_{12} = G_i (\varepsilon_{12}^0 + \varepsilon_{12}^\phi) = \sigma_{12} + \sigma_{12}^\phi$$

where $G_i$ is the shear modulus and is defined as

$$G_i = Y_i / 2(1 + \mu_i).$$ (39)
2.3. RESULTANT NORMAL FORCES AND MOMENTS

Generalized forces and moments for the i-th layer are illustrated in Figure 2. In this section, these forces and moments will be formulated by using the stresses and strains derived earlier.

The expression for calculating the resultant normal force of the layered shell, from Assumption 7 and equation (1), is

\[ N_{11} = \int_{\delta_0}^{\delta_1} \sigma_{11} \, d\alpha_3 = \sum_{i=1}^{n} \int_{\delta_i}^{\delta_{i+1}} \sigma_{11} \, d\alpha_3. \]  \hspace{1cm} (40)

Substituting equation (35) into equation (40) yields

\[ N_{11} = \sum_{i=1}^{n} \int_{\delta_i}^{\delta_{i+1}} (\sigma_{11} + \sigma'_{11}) \, d\alpha_3. \]  \hspace{1cm} (41)

Using equation (36) gives

\[ N_{11} = \sum_{i=1}^{n} \int_{\delta_i}^{\delta_{i+1}} \frac{Y_i}{1-\mu_i^2} \{ (\varepsilon_{11} + \mu_i \varepsilon_{22}) + [\varepsilon'_{11} + \mu_i (\varepsilon'_{22} + \varepsilon_{33})] \} \, d\alpha_3. \]  \hspace{1cm} (42)

Integrating over the thickness yields

\[ N_{11} = \sum_{i=1}^{n} \frac{Y_i}{1-\mu_i^2} \{ (\delta_{i+1} - \delta_i) [\varepsilon_{11}^{\mu_i} + \varepsilon_{11}^{\mu_i} + \mu_i (\varepsilon_{22}^{\mu_i} + \varepsilon_{33}^{\mu_i})] \\
+ \frac{(\delta_{i+1}^2 - \delta_i^2)}{2} [k_{11} + k_{11} + \mu_i (k_{22} + k_{22}')] \} + \int_{\delta_i}^{\delta_{i+1}} \mu_i \varepsilon_{33} \, d\alpha_3 \}. \]  \hspace{1cm} (43)

One can define

\[ \nabla_N(\alpha_1, \alpha_2) = \frac{Y_i \mu_i}{1-\mu_i^2} \int_{\delta_i}^{\delta_{i+1}} \varepsilon_{33} \, d\alpha_3, \quad K_i = \frac{Y_i h_i}{1-\mu_i^2}. \]  \hspace{1cm} (44, 45)

Note that \( (\delta_{i+1} - \delta_i) = h_i \), the thickness of the i-th layer. Applying equations (31), (32), (44), and (45) gives

\[ N_{11} = \sum_{i=1}^{n} \left[ K_i (\varepsilon_{11}^{\mu_i} + \mu_i \varepsilon_{33}^{\mu_i}) + \frac{Y_i (\delta_{i+1}^2 - \delta_i^2)}{2(1-\mu_i^2)} (k_{11} + \mu_i k_{22}') + \nabla_N \right]. \]  \hspace{1cm} (46)

Figure 2. Generalized forces and moments for the i-th layer shell. Note: subscript i is omitted for all forces and moments.
\[ N_{22} = \sum_{i=1}^{n} \left[ K_i \left( \epsilon_{12}^e + \mu_i \epsilon_{11}^e \right) + \frac{Y_i (\delta_{i+1}^2 - \delta_i^2)}{2(1 - \mu_i^2)} (k_{22}^e + \mu_i k_{11}^e) + \nabla N \right]. \]  

\[ N_{12} = \sum_{i=1}^{n} \left[ G_i (\delta_{i+1} - \delta_i) \epsilon_{12}^e + \frac{G_i (\delta_{i+1}^2 - \delta_i^2)}{2} k_{12}^e \right]. \]

The resultant moments of the layered shell can be evaluated in a similar manner (see equation (2)):

\[ M_{11} = \int_{\delta_1}^{\delta_n} \alpha_3 \sigma_{11} \, d\alpha_3 = \sum_{i=1}^{n} \int_{\delta_i}^{\delta_{i+1}} \alpha_3 \sigma_{11} \, d\alpha_3. \]

Using equations (33)-(35) gives

\[ M_{11} = \sum_{i=1}^{n} \left[ \frac{Y_i}{1 - \mu_i^2} \int_{\delta_i}^{\delta_{i+1}} \alpha_3 [\epsilon_{11}^e + \mu_i (\epsilon_{22}^e + \epsilon_{33}^e)] \, d\alpha_3 \right]. \]

Define

\[ \nabla_M (\alpha_1, \alpha_2) = \frac{Y_i}{1 - \mu_i^2} \int_{\delta_i}^{\delta_{i+1}} \alpha_3 \epsilon_{31} \, d\alpha_3. \]

Integrating over the thickness yields

\[ M_{11} = \sum_{i=1}^{n} \left\{ \frac{Y_i (\delta_{i+1}^2 - \delta_i^2)}{2(1 - \mu_i^2)} (\epsilon_{11}^e + \mu_i \epsilon_{22}^e) + \nabla_M + \frac{Y_i (\delta_{i+1}^2 - \delta_i^2)}{3(1 - \mu_i^2)} (k_{11}^e + \mu_i k_{22}^e) \right\}. \]

Similarly,

\[ M_{22} = \sum_{i=1}^{n} \left\{ \frac{Y_i (\delta_{i+1}^2 - \delta_i^2)}{2(1 - \mu_i^2)} (\epsilon_{22}^e + \mu_i \epsilon_{11}^e) + \nabla_M + \frac{Y_i (\delta_{i+1}^2 - \delta_i^2)}{3(1 - \mu_i^2)} (k_{22}^e + \mu_i k_{11}^e) \right\}, \]

\[ M_{12} = \sum_{i=1}^{n} \left\{ G_i \frac{(\delta_{i+1}^2 - \delta_i^2)}{2} \epsilon_{12}^e + G_i \frac{(\delta_{i+1}^2 - \delta_i^2)}{3} k_{12}^e \right\}. \]

Now the resultant normal forces, equations (46-48), and moments, equations (52-54), have been evaluated.

2.4. Modified Love Equations: Shell with Distributed Piezoelectric Control Layers

Hamilton's principle can now be applied to derive Love's equations for each layer [22, 24]. Because of its tedious and complicated derivation, only the final system equations are presented here. For detailed derivation, the reader may refer to the above two references. In the i-th layer the external forces in the principal directions are denoted by \( Q_{11}, Q_{22}, Q_{33}, \) and the Love equations will be evaluated in terms of these. To obtain the overall Love equations for the multi-layered shell, the corresponding equations are integrated over all the layers so that the resultant equations are in terms of the normal forces and moments for the entire shell. The modified Love equations for the i-th layer are

\[-\frac{\partial}{\partial \alpha_1} (N_{11} A_2) - \frac{\partial}{\partial \alpha_2} (N_{22} A_1) - N_{12} \frac{\partial A_1}{\partial \alpha_1} + N_{22} \frac{\partial A_2}{\partial \alpha_2} - A_1 A_2 \frac{Q_{11}}{R_1} + A_1 A_2 \rho \partial \tilde{u}_1 = A_1 A_2 Q_{11},\]

\[-\frac{\partial}{\partial \alpha_1} (N_{12} A_2) - \frac{\partial}{\partial \alpha_2} (N_{22} A_1) - N_{12} \frac{\partial A_2}{\partial \alpha_1} + N_{22} \frac{\partial A_1}{\partial \alpha_2} - A_1 A_2 \frac{Q_{22}}{R_2}.\]
\[ + A_1 A_2 \rho \lambda_2 u_2 = A_1 A_2, \quad Q_{21}, \] (56)
\[ - \frac{\partial}{\partial \alpha_1} (Q_{13}, A_1) - \frac{\partial}{\partial \alpha_2} (Q_{23}, A_1) + A_1 A_2 \left[ \frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} \right] + A_1 A_2 \rho \lambda_2 \bar{u}_2 = A_1 A_2 Q_{31}, \] (57)
where \( Q_{13} \) and \( Q_{23} \) are defined by
\[ (\alpha/\partial \alpha_1)(M_{11}, A_2) + (\alpha/\partial \alpha_2)(M_{12}, A_1) + M_{12} \alpha A_1/\partial \alpha_2 - M_{22} (\alpha A_2/\partial \alpha_1) - Q_{13}, A_1 A_2 = 0, \] (58)
\[ (\alpha/\partial \alpha_1)(M_{12}, A_2) + (\alpha/\partial \alpha_2)(M_{22}, A_1) + M_{12} \alpha A_2/\partial \alpha_1 - M_{22} (\alpha A_1/\partial \alpha_2) - Q_{23}, A_1 A_2 = 0. \] (59)

The method of obtaining the Love equations for the entire shell will be demonstrated on the first equation:
\[ \sum_{i=1}^{n} \left[ - \frac{\partial}{\partial \alpha_1} (N_{11}, A_2) - \frac{\partial}{\partial \alpha_2} (N_{21}, A_1) - N_{12} \frac{\alpha A_1}{\partial \alpha_2} + N_{22} \frac{\alpha A_2}{\partial \alpha_1} - A_1 A_2 \frac{Q_{13}}{R_1} + A_1 A_2 \rho \lambda_2 \bar{u}_1 \right] \]
\[ = \sum_{i=1}^{n} [A_1 A_2 Q_{i1}]. \] (60)

Applying equations (1) and (46)-(48) yields
\[ - \frac{\partial}{\partial \alpha_1} (N_{11}, A_2) - \frac{\partial}{\partial \alpha_2} (N_{21}, A_1) - N_{12} \frac{\alpha A_1}{\partial \alpha_2} + N_{22} \frac{\alpha A_2}{\partial \alpha_1} - A_1 A_2 \frac{Q_{13}}{R_1} + A_1 A_2 \rho \lambda_2 \bar{u}_1 = A_1 A_2 Q_{11}, \] (61)
where \( Q_{13} = \sum_{i=1}^{n} Q_{13}, \) and is obtained from
\[ \sum_{i=1}^{n} \left[ \frac{\partial}{\partial \alpha_1} (M_{11}, A_2) + \frac{\partial}{\partial \alpha_2} (M_{12}, A_1) + M_{12} \frac{\alpha A_1}{\partial \alpha_2} - M_{22} \frac{\alpha A_2}{\partial \alpha_1} - Q_{13}, A_1 A_2 \right] = 0. \] (62)

In this manner, the modified Love equations of motion for the entire multiple-layer shell are obtained as
\[ - \frac{\partial}{\partial \alpha_1} (N_{11}, A_2) - \frac{\partial}{\partial \alpha_2} (N_{21}, A_1) - N_{12} \frac{\alpha A_1}{\partial \alpha_2} + N_{22} \frac{\alpha A_2}{\partial \alpha_1} - A_1 A_2 \frac{Q_{13}}{R_1} \]
\[ + A_1 A_2 \rho \lambda_2 \bar{u}_1 = A_1 A_2 Q_{11}, \] (63)
\[ - \frac{\partial}{\partial \alpha_1} (N_{12}, A_2) - \frac{\partial}{\partial \alpha_2} (N_{22}, A_1) - N_{12} \frac{\alpha A_2}{\partial \alpha_1} + N_{22} \frac{\alpha A_1}{\partial \alpha_2} - A_1 A_2 \frac{Q_{23}}{R_2} \]
\[ + A_1 A_2 \rho \lambda_2 \bar{u}_2 = A_1 A_2 Q_{21}, \] (64)
\[ - \frac{\partial}{\partial \alpha_1} (Q_{13}, A_2) - \frac{\partial}{\partial \alpha_2} (Q_{23}, A_1) + A_1 A_2 \left[ \frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} \right] + A_1 A_2 \rho \lambda_2 \bar{u}_3 = A_1 A_2 Q_{31}, \] (65)
where \( Q_{13} \) and \( Q_{23} \) are defined by
\[ (\alpha/\partial \alpha_1)(M_{11}, A_2) + (\alpha/\partial \alpha_2)(M_{12}, A_1) + M_{12} \alpha A_1/\partial \alpha_2 - M_{22} (\alpha A_2/\partial \alpha_1) - Q_{13}, A_1 A_2 = 0, \] (66)
\[ (\alpha/\partial \alpha_1)(M_{12}, A_2) + (\alpha/\partial \alpha_2)(M_{22}, A_1) + M_{12} \alpha A_2/\partial \alpha_1 - M_{22} (\alpha A_1/\partial \alpha_2) - Q_{23}, A_1 A_2 = 0. \] (67)

The resultant forces and moments of the multi-layered thin shell are defined in equations (46)-(48) and (52)-(54): i.e.,
\[ M_{jk} = \sum_{i=1}^{n} M_{jk}, \quad j, k = 1, 2, \quad N_{jk} = \sum_{i=1}^{n} N_{jk}, \quad j, k = 1, 2, \] (68a, b)
\[ Q_1 = \sum_{i=1}^{n} Q_1, \quad j = 1, 2, 3, \quad Q_{k3} = \sum_{i=1}^{n} Q_{k3}, \quad k = 1, 2, \quad \rho h = \sum_{i=1}^{n} \rho h_i. \] (69, 70a, b)

The derived general equations of motion for the multi-layered thin shell can then be directly simplified to various geometries, such as cylinder, sphere, plate, beam, etc. The boundary conditions should be defined according to practical situations.
3. CASE STUDY: A CANTILEVER BEAM WITH A DISTRIBUTED POLYMERIC PIEZOELECTRIC ACTUATOR

Consider a Plexiglas beam with a distributed piezoelectric PVDF control layer glued onto its top surface. The PVDF layer is subjected to an electric potential, \( V(a_1, t) \), that is applied across its thickness. Due to the converse piezoelectric effect, an electromechanical control force will be generated in the PVDF layer [20]. The modified Love equations developed earlier can be used to obtain the equations of motion for this composite beam structure subjected to a localized piezoelectric forcing function. The frequency of the first mode and the damping ratio will be evaluated by theoretical, numerical, and experimental methods. Then the response of the beam with and without the active piezoelectric control force will be studied numerically and experimentally. The damping ratios obtained for various values of feedback gain will be compared.

3.1. REDUCTION OF THE MODIFIED LOVE EQUATIONS TO THE CASE OF A CANTILEVER BEAM WITH A PIEZOELECTRIC CONTROL LAYER

Assume that the effect of the applied voltage on the PVDF is to cause a longitudinal strain \( \varepsilon_{11} \) only [20]. Both layers are of the same width and length, in keeping with Assumption 2. The \( \alpha_1 \)-direction is along the length of the beam, the \( \alpha_2 \)-direction is along the width, and \( \alpha_3 \) is the transverse direction. Figure 3 illustrates the co-ordinate system and Figure 4 gives the cross-section. The subscripts 1 and 2 refer to the Plexiglas and the PVDF respectively. Variations in the \( \alpha_2 \)-direction are not considered, i.e. \( \delta(\cdot)/\delta \alpha_2 = 0 \). The neutral axis can be calculated by a standard moment analysis,

\[
\delta_b = \frac{Y_1 h_1^2 + Y_2 h_2^2 + 2 Y_1 h_1 h_2}{2(Y_1 h_1 + Y_2 h_2)}, \quad \delta_b = \frac{Y_1 h_1^2 - Y_2 h_2^2}{2(Y_1 h_1 - Y_2 h_2)}.
\]  

(71, 72)

For a cantilever beam, one can define

the radius of curvature, \( R_1 = R_2 = \infty \),

(73)

the fundamental form, \((ds)^2 = (d\alpha_1)^2 + (d\alpha_2)^2\)

(74)

the Lamé parameters, \( A_1 = A_2 = 1 \).

(75)

3.1.1. Plexiglas layer

This layer is not subjected to any localized forcing functions. Hence

\( \Delta_{11} = \Delta_{21} = \Delta_{33} = 0, \quad \beta_{11} = \beta_{22} = 0, \quad \varepsilon_{11} = \varepsilon_{22} = \varepsilon_{12} = \varepsilon_{33} = 0, \)

(76)

Figure 3. A cantilever beam coupled with a polymeric piezoelectric actuator. (Not to scale.)
\[ \beta_{11} = -\partial u_3/\partial \alpha_1, \quad \varepsilon_{11}^0 = \partial u_3/\partial \alpha_1, \quad k_{11} = -\partial^2 u_3/\partial \alpha_1^2. \]  

(76)

From equations (46)-(48), normal forces are defined:

\[ N_{111} = \int_{-h}^{h} \frac{Y_1}{1-\mu_1} [\varepsilon_{11}^0 + \alpha_3 k_{11}] d\alpha_3, \quad N_{221} = N_{12} = 0. \]  

(77, 78)

From equations (52)-(54), moments are defined as

\[ M_{111} = \int_{-h}^{h} \frac{Y_1}{1-\mu_1} [\varepsilon_{11}^0 + \alpha_3 k_{11}] \alpha_3 d\alpha_3, \quad M_{221} = M_{12} = 0. \]  

(79, 80)

Love's equations are then reduced to

\[ -\partial N_{111}/\partial \alpha_1 + \rho_1 h_1 \ddot{u}_1 = Q_{11}, \quad -\partial^2 M_{111}/\partial \alpha_1^2 + \rho_1 h_1 \ddot{u}_3 = Q_3. \]  

(81a, b)

### 3.1.2. Piezoelectric control layer

This layer is subjected to a localized forcing function, the control feedback voltage, which causes stresses and strains in the PVDF layer only (Assumption 5):

\[ \Delta_2 = \Delta_3 = 0, \quad \beta_2 = 0, \quad \beta_3 = -\partial u_3/\partial \alpha_1, \quad \varepsilon_{12}^0 = \partial u_3/\partial \alpha_1, \quad \varepsilon_{13}^0 = \partial \Delta_3/\partial \alpha_1. \]  

(82)

It is recognized that \( \varepsilon_{13}^0 \) is the (extra) longitudinal strain induced in the piezoelectric layer due to the applied voltage. This is given by

\[ \varepsilon_{13}^0 = d_{31} V(\alpha_1, t) = G^* d_{31} V^*(\alpha_1, t), \]  

(83)

\[ k_{13} = -\partial^2 u_3/\partial \alpha_1^2, \quad \varepsilon_{22}^0 = \varepsilon_{12}^0 = 0. \]  

(84)

where \( G^* \) is the feedback gain and \( V^* \) is an output voltage. Normal forces are defined:

\[ N_{112} = \int_{-h}^{h} \frac{Y_2}{1-\mu_2} [\varepsilon_{12}^0 + \varepsilon_{112}^0 + \alpha_3 k_{11}] d\alpha_3, \quad N_{222} = N_{12} = 0. \]  

(85)

Moments are calculated:

\[ M_{112} = \int_{-h}^{h} \frac{Y_2}{1-\mu_2} [\varepsilon_{12}^0 + \varepsilon_{112}^0 + \alpha_3 k_{11}] \alpha_3 d\alpha_3, \quad M_{222} = M_{12} = 0. \]  

(86)

Love's equations reduce to

\[ -\partial N_{111}/\partial \alpha_1 + \rho_2 h_2 \ddot{u}_1 = Q_{11}, \quad -\partial^2 M_{111}/\partial \alpha_1^2 + \rho_2 h_2 \ddot{u}_3 = Q_3. \]  

(87a, b)

### 3.1.3. Love equations of motion for the entire beam

Summing all normal forces and moments of each layer yields

\[ -(\partial/\partial \alpha_1)(N_{111} + N_{112}) + (\rho_1 h_1 + \rho_2 h_2) \ddot{u}_1 = Q_{11} + Q_{111}, \]  

(88a)

\[ -(\partial^2/\partial \alpha_1^2)(M_{111} + M_{112}) + (\rho_1 h_1 + \rho_2 h_2) \ddot{u}_3 = Q_3 + Q_{112}. \]  

(88b)
With the simplified notation described earlier, equations (46)-(48) and (52)-(54), for considering the transverse motion only, yield

$$-\dot{\alpha}^3 M_{11} + \dot{\alpha}^3 \dot{\psi}_3 = Q_3.$$  \hfill (89)

The moment of inertia terms about the "composite" neutral axis can be evaluated by using the parallel axis theorem:

$$I_1 = \frac{(wh_1^2/12)}{12} + wh_1[(h_1/2) - \delta_b], \quad I_2 = \frac{(wh_2^2/12)}{12} + wh_2[(h_1 + (h_2/2) - \delta_b)^2].$$  \hfill (90a, b)

Assuming \((1 - \mu^2) = (1 - \mu^2) = 1\) gives

$$wM_{11} = -YI(\dot{\alpha}^3 u_3 / \dot{\alpha}^3) + \nu V(\alpha_1, t),$$  \hfill (91a)

where

$$YI = Y_1 I_1 + Y_2 I_2,$$  \hfill (91b)

$$\nu = \frac{[d_1 Y_1 Y_2 h_1(h_1 + h_2)]/[2(Y_1 h_1 + Y_2 h_2)]}{\nu V(\alpha_1, t)}.$$  \hfill (91c)

Note that \(YI\) and \(\nu\) are constants for a given configuration. In this manner, the equation of motion for transverse vibrations is found to be

$$\left(\dot{\alpha}^3 / \dot{\alpha}^3 \right) \left[ YI \frac{\dot{\alpha}^3 u_3}{\dot{\alpha}^3} - \nu V(\alpha_1, t) \right] + wph\dot{u}_3 = wQ_3.$$  \hfill (92)

The second term in the parenthesis is interpreted as the moment induced by the feedback voltage, relative to the neutral axis of the two layered beam, opposing the beam motion. It is proportional to the feedback voltage. This equation is identical to that shown in reference [15] derived by a conventional method.

3.2. Active vibration control of a cantilever beam by using a PVDF control layer—Experimental and computational results

The response of a cantilever beam with a piezoelectric PVDF control layer and a miniaturometer to monitor its tip acceleration is considered here (see Figure 3). The accelerometer output is used in the feedback controls. The free responses of the PVDF-Plexiglas beam was observed experimentally and simulated numerically with and without feedback, and the results are compared here. The experimental set-up is illustrated in Figure 5; and material properties are listed in Table 1.

3.2.1. Free vibration without feedback

An initial displacement of 1 cm (0.3937 in) was applied to the tip to ensure the presence of the first mode, and the beam was allowed to snap back and oscillate. The period oscillation was used to determine the natural frequency. The natural frequency was also calculated analytically by

$$f_n = (1/2\pi)\left[3 Y_1 I_1 / (m_{p} + 0.23 m_i)\right]^{1/2} = 43.9 \text{ Hz}.$$  \hfill (93)

where the Young's modulus is assumed complex to include natural damping [2]. The natural frequency of the beam was found to be 43.8 Hz by laboratory experiments which compares well with a numerical and analytical solution of 43.9 Hz. A time history (theoretical) for zero feedback is presented in Figure 6 for future comparison.

The damping ratio \(\zeta\) was estimated by using the logarithmic decrement method, i.e.,

$$\zeta = \ln (y_n / y_{n+1}) / 2\pi(n).$$  \hfill (94)

Again, the initial damping ratio was estimated to be 2.501% by the numerical analysis [2] and 2.454% from laboratory experiments by the logarithmic decrement method. Thus, for the free oscillation without voltage feedback, these results compared favorably.
Figure 5. Apparatus of the active vibration suppression. (Not to scale.)

**Table 1**

<table>
<thead>
<tr>
<th>Properties</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y_1$</td>
<td>$3.447 \times 10^8 \text{ N/m}^2$ (modulus)</td>
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<tr>
<td>$\rho_1$</td>
<td>$1190 \text{ kg/m}^3$ (density)</td>
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<td>$l$</td>
<td>$6.985 \times 10^{-2} \text{ m}$ (length)</td>
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<tr>
<td>$w$</td>
<td>$6.35 \times 10^{-2} \text{ m}$ (width)</td>
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<tr>
<td>$h_1$</td>
<td>$1.5875 \times 10^{-3} \text{ m}$ (thickness)</td>
</tr>
<tr>
<td>$m_1$</td>
<td>$8.3792 \times 10^{-4} \text{ kg}$ (beam mass)</td>
</tr>
<tr>
<td>$I_s$</td>
<td>$2.117 \times 10^{-12} \text{ m}^4$ (wh/12)</td>
</tr>
<tr>
<td>$m_{\text{ac}}$</td>
<td>$6.5 \times 10^{-4} \text{ kg}$ (accelerometer mass)</td>
</tr>
<tr>
<td>$d_{31}$</td>
<td>$8 \times 10^{-22} \text{ C/N}$ (piezoelectric constant)</td>
</tr>
<tr>
<td>$h_2$</td>
<td>$40 \times 10^{-6} \text{ m}$ (thickness)</td>
</tr>
<tr>
<td>$Y_2$</td>
<td>$3.704 \times 10^8 \text{ N/m}^2$ (modulus)</td>
</tr>
</tbody>
</table>

† Subscript 1 for Plexiglas and 2 for PVDF.

Figure 6. Free response of the Plexiglas–PVDF beam without feedback.
3.2.2. Active, electromechanical vibration control

The beam response was studied for different amounts of feedback gain. The experimental feedback gain was easily obtained from the apparatus. The same range of values for the gain was implemented in the numerical analysis. An initial displacement of 1 cm (0.3937 in) was again applied to the tip of the cantilever beam and the free oscillation response recorded to estimate the damping ratio by using the logarithmic decrement method. (Note that the feedback is a velocity feedback; thus, damping is changed; see Figure 5.) The time histories (theoretical) for $C_{gain} = 0$ and 420 are presented in Figures 6 and 7, respectively. Note that $C_{gain}$ includes the effect of $d_{31}$, geometric configuration, material properties, accelerometer sensitivity, and feedback gain; see equations (83) and (91).

![Graph](image)

Figure 7. Time history of the beam at 420 feedback gain.

The experimentally obtained variations in the damping ratio with changes in the feedback gain can be compared to the values obtained from the numerical analysis. Table 2 summarizes damping ratio changes with various amounts of feedback gain.

The damping ratio versus the feedback gain is plotted in Figure 8. Linear regression lines were computed for both the experimental and numerical data sets. The regression line equations are as follows:

(i) numerical data,

$$\zeta = 2.501 \times 10^{-2} + 2.562 \times 10^{-5} \times C_{gain},$$

with a variance of $4.852 \times 10^{-6}$; (95)

(ii) experimental data,

$$\zeta = 2.865 \times 10^{-2} + 2.338 \times 10^{-5} \times C_{gain},$$

with a variance of $2.600 \times 10^{-6}$. (96)

The slope of the experimental data is within 9% of the slope of the numerical data. Also, the variance for both cases is very small, which indicates that the regression line fits the data very well. The slope can be improved by using a thicker PVDF actuator or increasing the moment arm in some way. Some aspects of the experiment could not be accounted for in the numerical analysis. The bond between the PVDF and the Plexiglas layers was not perfectly rigid and could lead to discrepancies between numerical and experimental values of damping. In general, the damping obtained experimentally can be expected to be a little higher than the values obtained numerically. Charge is sometimes lost to the atmosphere in the form of sparks, especially when the feedback voltage is high. A difference between the experimental and numerical curves is noticed, with the experimental values being higher than the numerical by an average of 9%. Experimentally, the increase in damping falls off for higher again. This can be explained by the charge
leakage that occurs at high feedback voltages which prevents any further increase in damping. This characteristic cannot be accounted for in the numerical analysis. Another point is that the regression curve for the numerical results shows a damping ratio of 0.025 for $C_{\text{gain}} = 0$. However, the regression curve for the experimental results shows $\zeta = 0.02865$ at $C_{\text{gain}} = 0$. This is because the linear regression line is an approximation and some variation around it is expected.

4. SUMMARY AND CONCLUSIONS

This paper presents a theoretical study of a multi-layered thin shell coupled with an active distributed electromechanical actuator(s)—polymeric piezoelectric polyvinylidene-fluoride (PVDF)—for distributed vibration controls. Dynamic equations of motion for the generalized multi-layered thin shell coupled with polymeric piezoelectrets were derived based on Love’s theory and Hamilton’s principle. Each layer of the shell can be a polymeric piezoelectric control layer subjected to a local control force to suppress the vibration of the shell. The control force is generated in the piezoelectric layer via the converse piezoelectric effect introduced by feedback voltages with various gains.

A cantilever beam coupled with the polymeric PVDF actuator was used as a case study to demonstrate the theory. The equation of motion of the beam was obtained by directly simplifying the derived multi-layered shell equations. Experimental model was also built and tested in laboratory to validate the results. Effectiveness of the active distributed piezoelectric vibration controller was evaluated. For the active control case, we found that numerically the variation of the damping ratio with the feedback gain was
approximated by $\zeta = 2 \cdot 501 \times 10^{-2} + 2 \cdot 562 \times 10^{-3} \times C_{gain}$ with a variance of $4 \cdot 852 \times 10^{-5}$. Experimentally we obtained $\zeta = 2 \cdot 868 \times 10^{-2} + 2 \cdot 338 \times 10^{-3} \times C_{gain}$ with a variance of $2 \cdot 474 \times 10^{-6}$. The slopes compare very favorably. The slope can be improved by using a thicker PVDF polymer or increasing the moment arm in some way. Possible reasons causing the deviation were also discussed.

The two aspects of piezoelectric PVDF, the direct piezoelectric effect applied to sensing and the converse piezoelectric effect applied to active vibration control, can be integrated to achieve both sensing and active control of the layered thin shells. The output of a PVDF accelerometer sensor (e.g., accelerometer in this case) provides a signal which can be integrated, phase-shifted, amplified and fed back to the PVDF control layer(s) to achieve a velocity feedback control. In this manner, the vibration of the thin shell can be continuously monitored and undesirable vibrations can be damped out before they become large enough to be harmful.

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APPENDIX: NOMENCLATURE

<table>
<thead>
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<th>Symbol</th>
<th>Description</th>
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<tr>
<td>$A_1, A_2$</td>
<td>Lamé parameters or fundamental form parameters</td>
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<tr>
<td>$d$</td>
<td>piezoelectric constant</td>
</tr>
<tr>
<td>$E$</td>
<td>electric field</td>
</tr>
<tr>
<td>$f$</td>
<td>frequency (Hz) = $\omega / 2\pi$</td>
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<td>$G$</td>
<td>bulk modulus</td>
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<tr>
<td>$G^*$</td>
<td>feedback gain</td>
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<td>$h$</td>
<td>a weighted average thickness of the multi-layered shell</td>
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<td>thickness of the $i$th layer ($=h_{i+1} - h_i$)</td>
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<td>Young's modulus</td>
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<tr>
<td>$\rho_i$</td>
<td>density of the $i$th layer</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>stress</td>
</tr>
<tr>
<td>$\omega$</td>
<td>radian frequency (rad/s), $= 2\pi f$</td>
</tr>
<tr>
<td>$\nabla_n, \nabla_m$</td>
<td>functions of $\alpha_1$ and $\alpha_2$ defined in Equations (45) and (51) respectively</td>
</tr>
</tbody>
</table>
Superscripts

τ “total”, indicates that the term includes the effects of the induced (localized) forcing functions

' "prime", indicates that the term expresses the particular effect of the induced (localized) forcing functions

Subscripts

b bottom-most or base
exc excitation
fb feedback
i indicates that the term applies to the ith layer only
r resultant
t topmost