DYNAMIC ANALYSIS AND PASSIVE CONTROL OF VISCOELASTICALLY DAMPED NONLINEAR DYNAMIC CONTACTS

Horn S. TZOU
Department of Mechanical Engineering, University of Kentucky, Lexington, KY 40506-0046, U.S.A.

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Abstract. This paper presents an analytical and finite element study on nonlinear contact dynamics and controls. Nonlinear dynamic contacts between eccentrically supported masses and simply supported beams are studied. Passive control of the dynamic contacts using viscoelastic dampers is also proposed and evaluated. A nonlinear contact finite element is modeled by a set of nonlinear stiffness and damping polynomial functions; and a nonlinear viscoelastic contact finite element is modeled by a Standard Linear Model with frequency-dependent nonlinear stiffness and damping functions. Analyses show that the dynamic contact force increases as the initial gap increases. Application of viscoelastic dampers can effectively reduce contact loads and prevent dynamic contacts. A simple design equation is also proposed.

Introduction

A complex dynamic system with multiple masses eccentrically supported between two simply supported beams experienced dynamic contacts imposed by in-plane base excitations. It was observed that the masses oscillated freely between the beams at relatively low-amplitude base excitations. However, the masses contacted the beams and bounced between the two beams when the relative displacement exceeded an initial gap. The dynamic contacts introduced significant contact loads and resulted in system failure. This paper is intended to study contact dynamics and to propose design modification controlling the contacts. It presents a mathematical and computational study of these dynamic contact phenomena. A passive control technique using viscoelastic dampers is also proposed and evaluated.

Due to the complexity of the system dynamics, the theoretical equations of a dynamic contact model are difficult to solve. Thus, a finite element method is used in this study. Since dynamic contact and viscoelastic damping are nonlinear (characteristics change with respect to time and state) in nature, a nonlinear finite element technique needs to be developed in the system modeling and analysis. In general, there are two fundamental techniques in the nonlinear finite element analysis: (1) the reformation of system matricies method, in which the system matrices need to be updated according to changes of the state, and an iteration procedure is usually employed to balance the residual loads [2,10]; and (2) the pseudo-force approximation method, in which an equivalent load derived from the nonlinearity is calculated as a pseudo-nonlinear force and is added to a linear force vector [5,6,9]. In either case, computation is very time-consuming, so numerical methods improving computation efficiency are usually used.

In this study, a nonlinear contact element is implemented in a Nonlinear Dynamic Analysis Finite Element Program (NDAFEPEP), developed by Tzou [7], in which the element is modeled by a set of nonlinear stiffness and damping polynomial functions. The element can also be used to represent either an active or a passive controller by specifying various combinations of stiffness and damping functions.
Viscoelastic dampers are very popular and widely used in vibration isolation and control. In general, they are introduced into the system in a passive way, i.e., a motion-activated passive dynamic controller. In this study, the viscoelastic damper is modeled by a Standard Linear Model [1,11] and implemented as a viscoelastic element in the NDAFEP.

In the finite element analysis, nonlinear forces respectively derived from the contact element and the viscoelastic element are transferred to the right side of the system equations as pseudo-force components, and they are directly added to a linear excitation force vector in the numerical time-domain integration [1,9]. Thus, reforming system matrices due to change of the nonlinearity is not required in the analysis. Due to the nature of nonlinear dynamic problems, a small integration-time step is required to ensure numerical stability. Dynamic responses of the system are calculated by a direct integration algorithm—the Wilson-θ Method [10]—with minor modifications to accommodate the pseudo-forces derived from the nonlinearity [1,9]. The Wilson-θ method applies a linear acceleration algorithm. Thus, the pseudo-force can only be determined by an extrapolation technique. Stricklin found that a linear extrapolation is sufficient to define the load [6]. Guyan’s reduction scheme [4] is also implemented in the NDAFEP to improve the computation efficiency.

In this paper, a theoretical derivation of a general multi-mass contact model will be presented first. The system equation will be extended to include viscoelastic control forces when viscoelastic dampers are introduced. Finite element modeling of dynamic contacts and viscoelastic dampers will also be discussed. Structural dynamics of a physical system with or without viscoelastic dampers will be studied and compared.

Derivation of the nonlinear equation of motion

In this section, system dynamic equations for a general dynamic contact model with multiple independent spring–mass–damper subsystems (Fig. 1) will be derived.

Each spring–mass–damper system is regarded as a unit system. A unit system $i$ has a mass $m_i$ (moving in translational direction $x_i$), a linear spring $k_i$, a linear damper $c_i$, a nonlinear spring $k_i^N$, and a nonlinear damper $c_i^N$. Dynamic contact is defined in a way that the relative displacement, $x_i - x_{i+1}$, is greater than an initial gap $x_0$, from which the nonlinear spring and damper start activating. Dynamic contact of adjacent unit systems and its effect on overall system response will be studied. Passive dynamic control using viscoelastic dampers will also be carried out in this study.

Fig. 1. A dynamic contact model with $n$ sub-systems separated by gaps.
The equation of motion of the dynamic system can be derived from Lagrange’s equation using general energy expressions:

\[
\begin{bmatrix}
m_1 & 0 & 0 & 0 \\
0 & m_2 & 0 & 0 \\
0 & 0 & m_3 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & m_n \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n \\
\end{bmatrix}
= 
\begin{bmatrix}
c_1^* & -c_1^*S(\lambda_1) & 0 & 0 \\
-c_1^*S(\lambda_1) & c_2^* & -c_2^*S(\lambda_2) & 0 \\
0 & -c_2^*S(\lambda_2) & c_3^* & -c_3^*S(\lambda_3) \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & c_n^* \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n \\
\end{bmatrix}
+ 
\begin{bmatrix}
k_1^* & -k_1^*S(\lambda_1) & 0 & 0 \\
-k_1^*S(\lambda_1) & k_2^* & -k_2^*S(\lambda_2) & 0 \\
0 & -k_2^*S(\lambda_2) & k_3^* & -k_3^*S(\lambda_3) \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & k_n^* \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n \\
\end{bmatrix}
\]

where \( S(\lambda_i) \) is a step function (i.e., \( S(\lambda_i) = 0 \) when \( \lambda_i < \gamma \), and \( S(\lambda_i) = 1 \) when \( \lambda_i \geq \gamma \)), \( c_i^* = c_i + c_i^N S(\lambda_i) + c_i^N S(\lambda_i - 1) \) \( (c_0 = 0) \), and \( k_i^* = k_i + k_i^N S(\lambda_i) \) \( (k_0 = 0) \), and \( k_n^* = k_n + k_n^N S(\lambda_{n-1}) \). Simply,

\[
[M] \cdot \{ \ddot{x} \} + [C] \cdot \{ \dot{x} \} + [K] \cdot \{ x \} = \{ F(t) \} + \{ F^N(t) \}.
\]

where \([M]\) is the system mass matrix; \([C]\) and \([K]\) are the nonlinear system damping and stiffness matrices; \{\ddot{x}\}, \{\dot{x}\}, and \{x\} are the acceleration, velocity, and displacement vectors; \{F(t)\} is the external excitation force vector; and \{F^N(t)\} is the nonlinear force vector. It should be noted that the system equation of motion, eqn. (1), can be decoupled and linearized when \( \kappa \) goes to infinity.

The pseudo-force method

The nonlinear system matrices can be divided into linear and nonlinear components as

\[
[C] = [C]^L + [C]^N
\]
\[
[K] = [K]^L + [K]^N
\]

in which \([C]^L\) and \([K]^L\) are the linear components, and \([C]^N\) and \([K]^N\) are the nonlinear components. Substituting eqns. (3) and (4) into eqn. (2) and moving all nonlinear terms to the right side of the equation yield

\[
[M] \cdot \{ \ddot{x} \} + ([C]^N) \cdot \{ \dot{x} \} + ([K]^L) \cdot \{ x \} = \{-[C]^N\} \cdot \{ \ddot{x} \} + \{-[K]^N\} \cdot \{ \dot{x} \} + \{ F(t) \} + \{ F^N(t) \}
\]
force $F_c(t)$ derived from the viscoelastic damper can be calculated by [8,9]

$$\{ F_c(t) \} = [C'(\delta \dot{x}')] \cdot \{ \delta \dot{x}' \} \cdot \{ S(\varphi) \} + [K'(\delta x')] \cdot \{ (\delta x') - (\epsilon) \} \cdot \{ S(\varphi) \}$$

(16)

where $[C'(\delta \dot{x}')]$ and $[K'(\delta x')]$ are the equivalent viscoelastic damping and stiffness matrices, respectively; $(\epsilon)$ is the gap vector; $S(\varphi)$ is the step function ($\varphi_t = x_t - x_{t+1} - \epsilon$); $(\delta \dot{x}')$ and $(\delta x')$ are the relative velocity and relative displacement vectors as defined in eqn. (12).

The control force $\{ F_c(t) \}$ resulting from the viscoelastic damper is transferred to the right side of the system equation as a control force vector:

$$\{ M \} \cdot \{ \ddot{x} \} + [C] \cdot \{ \dot{x} \} + [K] \cdot \{ x \} = \{ F(t) \} + \{ F_c(t) \} + \{ F_c(t) \}$$

(17)

Substituting eqns. (10), (11), and (15) into eqn. (17) yields a general equation for the dynamic contact model with viscoelastic dampers:

$$\{ M \} \cdot \{ \ddot{x} \} + [C] \cdot \{ \dot{x} \} + [K] \cdot \{ x \} = \{ F(t) \} + \sum_{i=1}^{n} c_i \delta \dot{x}_i^{(i-1)} \cdot \{ (\ddot{x}_i) - (\dot{x}_i) \} \cdot \{ S(\lambda) \}$$

$$+ \left[ \sum_{i=1}^{n} k_i \delta x_i^{(i-1)} \right] \cdot \{ (x_i) - (x_j) - (\lambda) \} \cdot \{ S(\lambda) \}$$

$$+ \left[ \frac{c_i k_{2i}}{k_{2i} + (\omega \varepsilon_i)^2} \right] \cdot \{ (\dot{x}_i') - (\dot{x}_j') \} \cdot \{ S(\varphi) \}$$

$$+ \left[ k_{v1} + \frac{k_{v2}(\omega \varepsilon_v)^2}{k_{v2} + (\omega \varepsilon_v)^2} \right] \cdot \{ (x_i') - (x_j') - (\epsilon) \} \cdot \{ S(\varphi) \}$$

(18)

The linear damping matrix of the model is usually assumed to be proportional to the stiffness and mass matrices by Rayleigh's coefficients $\alpha$ and $\beta$ in the finite element analysis,

$$[C^L] = \alpha \cdot [M] + \beta \cdot [K]^L$$

(19)

The damping ratio of the $r$th mode of the system is $\xi_r = (\alpha + \beta \omega_r^2) / 2 \omega_r$, where $\omega_r$ is the natural frequency of the $r$th mode.

**Case study**

A dynamic system consisting of five concentrated masses loosely hung between two simply supported beams is shown in Fig. 4. The masses are located at the lower two thirds of the
structure, and each mass weighs 1,480 pounds in the model. The side beam used in the model has a cross-section area \( A = 9.26 \text{ in}^2 \), a flexural rigidity \( EI = (2.9 \times 10^7)(6.47) \text{ lb-in}^2 \), and a damping ratio \( \xi = 2.0\% \). The initial gap \( x \) between the masses and the beam varies from 0 to 6 inches. The nonlinear contact element has a stiffness function \( k_{nr} = 5.4 \times 10^{-3} + 1.99 \times 10^4 \cdot \delta g \cdot S(\lambda_n) \text{ lb-in} \) and a damping function \( c_{nr} = 85 \cdot S(\lambda_n) \text{ lb-s/in} \). The \( k_1, 5.4 \times 10^{-3} \), represents the uncontact status, and it is also used to support the concentrated masses in the model to avoid numerical problems. The soft spring \( k_1 \) is \( 7.36 \times 10^{-6} \) of the equivalent beam stiffness and \( 2.72 \times 10^{-7} \) of the second nonlinear term.

Eigenvalue analysis

In this section, system natural frequencies and mode shapes are evaluated. The natural frequencies are summarized in Table 1. The analytical equation for solving the mass/soft

<table>
<thead>
<tr>
<th>Mode</th>
<th>Analytical solution (Hz)</th>
<th>NDAFEP Constant [M]</th>
<th>Lumped [M]</th>
</tr>
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<tbody>
<tr>
<td>1-5</td>
<td>( 8.451 \times 10^{-3} )</td>
<td>( 8.451 \times 10^{-3} )</td>
<td>( 8.451 \times 10^{-3} )</td>
</tr>
<tr>
<td>6-7</td>
<td>15.2040</td>
<td>15.2090</td>
<td>14.8711</td>
</tr>
<tr>
<td>8-9</td>
<td>60.8469</td>
<td>61.7302</td>
<td>63.9649</td>
</tr>
</tbody>
</table>
amplitude and frequency of the structural vibration determine the first component; and those of the contact oscillation determine the second component. When the beam becomes stiffer, both the contact frequency and the contact force increase.

**Viscoelastically damped vibration analysis**

Passive viscoelastic dampers introduced between the mass and the beam can soften the dynamic contact and provide additional damping to the system. Figure 10 shows a simplified finite element model with viscoelastic dampers. (The original model is defined in Fig. 4.) Various viscoelastic dampers (with different primary stiffness $k_{v1}$, secondary stiffness $k_{v2}$, and damping $c_{v}$) were tested using the finite element model with a 6-in. gap (Figs. 4 and 10). The secondary stiffness of the viscoelastic damper is assumed to be three times that of the primary stiffness for a damper made of rubber. The damping $c_{v}$ is determined by $c_{v} = 2\zeta_0 \sqrt{k_{v1} m}$ where $m$ is the applied concentrated mass and $\zeta_0$ is the damping ratio assumed to be 20% for the viscoelastic rubber damper.

The dynamic responses of the third mass with two adjacent beam nodes in the finite element model with different viscoelastic dampers ($k_{v1} = 0.0, 50.0, 100.0, 150.0, \text{and } 200.0 \text{ lb/in}$) and subjected to 0.5 Hz and 0.4 g base excitation are presented in Figs. 9 and 11-14. In these plots only the third mass motion and the two adjacent nodes on the side beams are presented.

The responses of the original system when $k_{v1} = 0.0$ (Fig. 9) show severe dynamic contact phenomena similar to some other figures with different initial gaps. When the primary stiffness increases to 50 lb/in, the contact is significantly reduced. As the $k_{v1}$ keeps increasing, the contact is eventually prevented. The equivalent loads on the beam applied by either direct contact and/or indirect compression of the viscoelastic damper are also estimated.

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![Fig. 10. The dynamic system with viscoelastic dampers.](image-url)
of the viscoelastic link damper, the maximum displacement of the third mass, the contact status, and the contact beam load are listed.

When \(k_{\nu} = 100 \text{ lb/m} \), mass contact is prevented and the minimum beam load is observed (Table 3). When the primary stiffness of the viscoelastic damper increases, the constraint applied to the beam and the mass results in increasing the absolute motion of the mass and further increasing the beam load. Thus, the best viscoelastic damper should be rigid enough to prevent the contact and be soft enough not to constrain the mass motion.

Viscoelastic damper design equation

The above study has shown that viscoelastic dampers can effectively prevent dynamic contacts and can also significantly reduce dynamic loads. A design principle for estimating required viscoelastic dampers is also proposed.

It is assumed that \(n\) modal responses are involved in the structural vibration. The modal participation factors \(\Psi_i\) can be used to estimate the amplitude of each mode. The most conservative estimation would be the summation of each modal response participating in the motion. Equation (20) can be used to estimate the required stiffness \(K^*\) and damping \(C^*\) for the viscoelastic damper:

\[
\lambda = \sum_{i=1}^{n} \left\{ \frac{\Psi_i M Y_i \omega_i^2}{\left[ (K^*-M\omega_i^2) + (C^*\omega_i)^2 \right]^{1/2}} \right\}
\]

where \(M\) is the mass; \(Y\) and \(\omega\) are the vibration amplitude and frequency, respectively; and \(\lambda\) is the design gap between two sub-systems. In practical applications, several iterations may be required to estimate appropriate viscoelastic dampers. It should also be noted that if there is a temperature fluctuation, designers should refer to temperature-related material properties.

Conclusions

In this paper, an analytical and finite element study on dynamic contacts and viscoelastic controls of the contacts were presented. A nonlinear contact element and a nonlinear viscoelastic element using a pseudo-force approximation technique were developed and evaluated. The major advantage of the technique is that all nonlinear terms are transferred to the right side of the system equation and are directly added to a linear force vector. Thus, homogeneous system matrices are always constant during the time-domain integration. The nonlinear contact element was represented by a combination of nonlinear stiffness and damping polynomial functions. The viscoelastic element was modeled by a Standard Linear Model with a complex stiffness which provided an equivalent stiffness and an equivalent damping function. The dynamic response was determined by a direct integration technique—the Wilson-\(\theta\) method modified to accommodate viscoelastic dampers and dynamic contacts.

A dynamic system with five masses eccentrically hung between two simply supported beams with various gaps was studied to evaluate the dynamic contacts. The study shows that the dynamic load consists of two components resulting from imposed motion and contact oscillation, respectively. The amplitude and frequency of the base motion determine the first component; and those of the contact oscillation determine the second component. When the beam becomes stiffer, both the contact frequency and the contact force increase. It is also noted
that when the initial gap increases, the force associated with the imposed motion decreases; and
that associated with the contact oscillation increases.

In order to reduce contact force and to prevent dynamic contacts, viscoelastic dampers
linking masses and side beams were introduced into the system. The study shows the
effectiveness of the viscoelastic dampers in preventing dynamic contacts and reducing dynamic
loads. A simple design equation for selecting viscoelastic dampers was also proposed in the
study.

List of symbols

<table>
<thead>
<tr>
<th></th>
<th>Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>( )</td>
<td>Vector</td>
</tr>
<tr>
<td>$A$</td>
<td>Cross section area</td>
</tr>
<tr>
<td>$c_i$</td>
<td>Damping coefficient</td>
</tr>
<tr>
<td>$c_{ij}$</td>
<td>Nonlinear damper for the $i$th unit system</td>
</tr>
<tr>
<td>$C'$</td>
<td>Damping function of the nonlinear element</td>
</tr>
<tr>
<td>$C_{ij}$</td>
<td>Damping matrix of the nonlinear element</td>
</tr>
<tr>
<td>$c_v$</td>
<td>Damping of viscoelastic damper</td>
</tr>
<tr>
<td>$C^N$</td>
<td>Equivalent viscoelastic damping function</td>
</tr>
<tr>
<td>$[C]$</td>
<td>System damping: $[C] = [C]^L + [C]^N$</td>
</tr>
<tr>
<td>$[C]^L$</td>
<td>Linear damping matrix</td>
</tr>
<tr>
<td>$[C]^N$</td>
<td>Nonlinear damping matrix</td>
</tr>
<tr>
<td>$D$</td>
<td>Energy dissipation: $D = D^L + D^N$</td>
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<td>Linear term of $D$</td>
</tr>
<tr>
<td>$D^N$</td>
<td>Nonlinear term of $D$</td>
</tr>
<tr>
<td>$E I$</td>
<td>Flexural rigidity</td>
</tr>
<tr>
<td>$f_n$</td>
<td>Natural frequency in Hertz</td>
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<tr>
<td>$F_0$</td>
<td>Amplitude of excitation</td>
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<td>${F(t)}$</td>
<td>Linear force vector</td>
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<tr>
<td>${F_n(t)}$</td>
<td>Pseudo force vector derived from nonlinear element</td>
</tr>
<tr>
<td>${F^N(t)}$</td>
<td>Nonlinear force vector</td>
</tr>
<tr>
<td>${F_v(t)}$</td>
<td>Pseudo force vector derived from viscoelastic damper</td>
</tr>
<tr>
<td>$k_i$</td>
<td>Stiffness coefficient</td>
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<td>$k_v$</td>
<td>Nonlinear stiffness for the $i$th unit system</td>
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<tr>
<td>$k_{ij}$</td>
<td>Stiffness function of the nonlinear element</td>
</tr>
<tr>
<td>$[K]$</td>
<td>Stiffness matrix of the nonlinear element</td>
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<tr>
<td>$K^N$</td>
<td>Primary stiffness of viscoelastic damper</td>
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<tr>
<td>$K_{v1}$</td>
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<td>Unit step function vector</td>
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<td>Kinetic energy</td>
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<td>$U$</td>
<td>Potential energy $U = U^L + U^N$</td>
</tr>
<tr>
<td>$U^L$</td>
<td>Linear potential energy</td>
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Acknowledgment

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References