Source Excitations in FEM

- In order to simulate real problems necessary for engineering design using the FEM, we need to be able to represent physical sources

- The source model required is often driven by the parameterization of the device under test that we are trying to extract.

- How the parameterization is extracted also dictates the level of accuracy required of the source model:
  - Exact representation of a true physical source
  - Approximate representation
    - Non-ideal nature of source is de-embedded
  - Non-physical source
    - Post-processing of data renders exact source un-necessary

Source Models

- Impressed current source
  - Discrete point-dipole
    - This can only be approximated in a FEM simulation
  - Current density
    - Line current (again, approximated)
    - Surface current
    - Volume current
- Plane Wave source
  - Plane wave injected into problem domain on a boundary
  - Exterior coupling (e.g., FE-BI)
- Waveguide mode excitation
  - Approximate, aperture coupled
  - Exact modal excitation (Exterior coupling problem)
- Discrete Lumped source model
  - Approximate Thevenin/Norton source model
  - Port mode (scattering parameter extraction)
  - Transmission line source model (approximate 1D modal excitation)

Impressed Current Density

- Recall the wave equation derived with impressed current densities:

$$\int_{\Omega}\left(\nabla\times\vec{E}^{a}\cdot\frac{1}{\mu_r}\nabla\times\vec{E}^{tot}-k_0^2\vec{E}^{a}\cdot\varepsilon_r\vec{E}^{tot}\right)d\Omega-\int_{\Gamma}\vec{E}^{a}\times\frac{1}{\mu_r}\nabla\times\vec{E}^{tot}\cdot\hat{n}ds=$$
$$-jk_o\eta_o\int_{\Omega_j}\vec{E}^{a}\cdot\vec{J}^{imp}d\Omega-\int_{\Omega_m}\vec{E}^{a}\cdot\nabla\times\frac{1}{\mu_r}\vec{M}^{imp}d\Omega$$

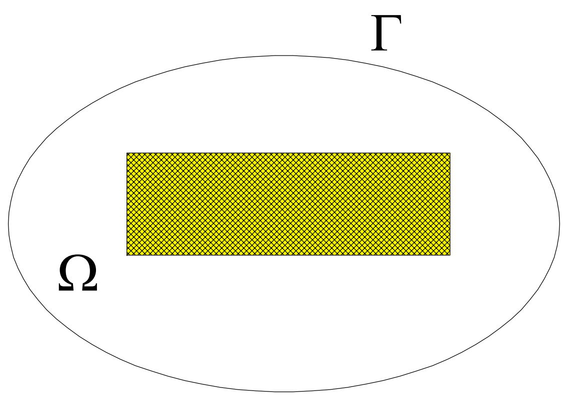
  - The impressed current densities are assumed to be distributed over finite volumes Ω_j and Ω_m

- Assume that there is an electric current density that is reduced to a surface. Then, this expression can be reduced to:

$$\int_{\Omega}\left(\nabla\times\vec{E}^{a}\cdot\frac{1}{\mu_r}\nabla\times\vec{E}^{tot}-k_0^2\vec{E}^{a}\cdot\varepsilon_r\vec{E}^{tot}\right)d\Omega-\int_{\Gamma}\vec{E}^{a}\times\frac{1}{\mu_r}\nabla\times\vec{E}^{tot}\cdot\hat{n}ds=-jk_o\eta_o\int_{\Gamma_j}\vec{E}^{a}\cdot\vec{J}^{imp}ds$$

Plane Wave Source Injection

- Assume that you have a finite dimensional object under test that is illuminated by a plane wave source
- The object lies completely within the domain Ω, bound by Γ
    - o The object under test is assumed to be a heterogeneous composition of conductors and penetrable materials



- The fields modeled via the FEM method are the total field intensities
    - o The total fields satisfy the expected boundary conditions on material/conductor surfaces

$$\int_{\Omega}\left(\nabla\times\vec{E}^{a}\cdot\frac{1}{\mu_r}\nabla\times\vec{E}^{tot}-k_0^2\vec{E}^{a}\cdot\varepsilon_r\vec{E}^{tot}\right)d\Omega-\int_{\Gamma}\vec{E}^{a}\times\frac{1}{\mu_r}\nabla\times\vec{E}^{tot}\cdot\hat{n}ds=0$$

Scattered Field Formulation

- The total field can be expressed as a superposition of the incident and scattered fields:
  - $\vec{E}^{tot} = \vec{E}^{inc} + \vec{E}^{scat}, \quad \vec{H}^{tot} = \vec{H}^{inc} + \vec{H}^{scat}$
  - Where, $\vec{E}^{inc}, \vec{H}^{inc}$ are the incident plane wave which are propagating through the homogeneous free space in the *absence* of the of the object under test.
  - The incident electric field satisfies the weak-form equation:

$$\int_{\Omega} \left(\nabla \times \vec{E}^{a} \cdot \nabla \times \vec{E}^{inc} - k_0^2 \vec{E}^{a} \cdot \vec{E}^{inc} \right) d\Omega - \int_{\Gamma} \vec{E}^{a} \times \nabla \times \vec{E}^{inc} \cdot \hat{n} ds = 0$$

  - Expanding the total field as a function of the incident and scattered fields:

$$\int_{\Omega} \left(\nabla \times \vec{E}^{a} \cdot \frac{1}{\mu_r} \nabla \times \left(\vec{E}^{inc} + \vec{E}^{scat} \right) - k_0^2 \vec{E}^{a} \cdot \varepsilon_r \left(\vec{E}^{inc} + \vec{E}^{scat} \right) \right) d\Omega$$

$$- \int_{\Gamma} \vec{E}^{a} \times \frac{1}{\mu_r} \nabla \times \left(\vec{E}^{inc} + \vec{E}^{scat} \right) \cdot \hat{n} ds = 0$$

  - The incident field equation is subtracted from this, leading to:

$$\int_\Omega \left(\nabla \times \vec{E}^a \cdot \frac{1}{\mu_r} \nabla \times \vec{E}^{scat} - k_0^2 \vec{E}^a \cdot \varepsilon_r \vec{E}^{scat} \right) d\Omega + jk_o\eta_o \int_\Gamma \vec{E}^a \times \vec{H}^{scat} \cdot \hat{n} ds$$

$$= \int_\Omega \left(\nabla \times \vec{E}^a \cdot \left(1 - \frac{1}{\mu_r} \right) \nabla \times \vec{E}^{inc} - k_0^2 \vec{E}^a \cdot (1-\varepsilon_r) \vec{E}^{inc} \right) d\Omega + jk_o\eta_o \int_\Gamma \vec{E}^a \times (\mu_r - 1) \vec{H}^{inc} \cdot \hat{n} ds$$

- Note that the excitation appears in the form of volume current sources in material regions
- Notes on Γ:
  - Near the exterior boundary, typically μ_r = 1, and the boundary term is 0
    - Exception can be for layered media of infinite planar slabs.
      - Boundary term included, and the scattered field requires correct ABC
  - PEC surfaces:
    - Dirichlet boundary condition:

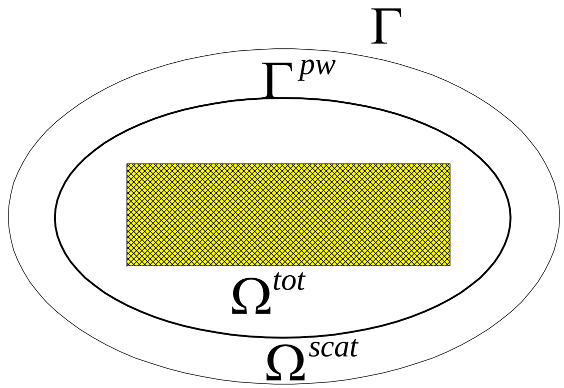
$$\hat{n} \times \vec{E}^{scat} = -\hat{n} \times \vec{E}^{inc}$$

  - PMC surfaces:
    - Neuman boundary condition:

$$\hat{n} \times \vec{H}^{scat} = -\hat{n} \times \vec{H}^{inc}$$

$$\therefore\ jk_o\eta_o \int_\Gamma \vec{E}^a \times \vec{H}^{scat} \cdot \hat{n} ds - jk_o\eta_o \int_\Gamma \vec{E}^a \times (\mu_r - 1) \vec{H}^{inc} \cdot \hat{n} ds = -jk_o\eta_o \int_\Gamma \vec{E}^a \times \mu_r \vec{H}^{inc} \cdot \hat{n} ds$$

Total-Field/Scattered-Field Formulation



- The domain Ω can be broken up into two regions, a scattered field region, and a total field region. These are separated by the boundary Γ^{pw}.
  - In the total field region:

$$\int_{\Omega^{tot}} \left(\nabla \times \vec{E}^{a} \cdot \frac{1}{\mu_r} \nabla \times \vec{E}^{tot} - k_0^2 \vec{E}^{a} \cdot \varepsilon_r \vec{E}^{tot} \right) d\Omega - \int_{\Gamma^{pw}} \vec{E}^{a} \times \nabla \times \vec{E}^{tot} \cdot \hat{n} ds - \int_{\Gamma^{obj}} \vec{E}^{a} \times \frac{1}{\mu_r} \nabla \times \vec{E}^{tot} \cdot \hat{n} ds = 0$$

  - In the scattered field region:

$$\int_{\Omega^{scat}} \left(\nabla \times \vec{E}^{a} \cdot \nabla \times \vec{E}^{scat} - k_0^2 \vec{E}^{a} \cdot \vec{E}^{scat} \right) d\Omega - \int_{\Gamma^{pw}} \vec{E}^{a} \times \nabla \times \vec{E}^{scat} \cdot \hat{n} ds - \int_{\Gamma} \vec{E}^{a} \times \nabla \times \vec{E}^{scat} \cdot \hat{n} ds = 0$$

- On the boundary Γ^{pw}, we must choose either a scattered field, or a total field as the unknown. Here, we will choose the scattered field. Thus, on Γ^{pw}
  - $\vec{E}^{tot} = \vec{E}^{scat} + \vec{E}^{inc}$, $\vec{H}^{tot} = \vec{H}^{scat} + \vec{H}^{inc}$

- This leads to:

$$\int_{\Omega^{tot}} \left(\nabla\times\vec{E}^a \cdot \frac{1}{\mu_r} \nabla\times\vec{E}^{tot} - k_0^2 \vec{E}^a \cdot \varepsilon_r \vec{E}^{tot} \right) d\Omega +$$

$$\int_{\Omega^{pw}} \left(\nabla\times\vec{E}^a \cdot \nabla\times\vec{E}^{scat} - k_0^2 \vec{E}^a \cdot \vec{E}^{scat} \right) d\Omega -$$

$$\int_{\Gamma^{pw}} \vec{E}^a \times\nabla\times\vec{E}^{scat} \cdot \hat{n} ds - \int_{\Gamma^{obj}} \vec{E}^a \times \frac{1}{\mu_r} \nabla\times\vec{E}^{tot} \cdot \hat{n} ds =$$

$$-\int_{\Omega^{pw}} \left(\nabla\times\vec{E}^a \cdot \nabla\times\vec{E}^{inc} - k_0^2 \vec{E}^a \cdot \vec{E}^{inc} \right) d\Omega - jk_o\eta_o \int_{\Gamma^{pw}} \vec{E}^a \times \vec{H}^{inc} \cdot \hat{n} ds$$

- Where, the integral over Ω^{pw} implies the integral inside Ω^{tot} due to basis associated with sub-topologies (i.e., edges or faces) on Γ^{pw}.
- Also, note that the boundary integral $\int_{\Gamma^{pw}} \vec{E}^a \times\nabla\times\vec{E}^{scat} \cdot \hat{n} ds$ due to complementary normals on either side of Γ^{pw}.

- To evaluate the incident field terms, we can treat the boundary as a Dirichlet boundary, expand the incident electric field via the curl-conforming basis on Γ^{pw}

$$\vec{E}^{inc}\Big|_{\Gamma^{pw}} = \sum_{i=1}^{N_{pw}} c_i^{inc}\vec{W}_i$$

- This can be formulated into a linear system of equations as:

$$\int_{\Gamma^{pw}} \vec{W}_j \cdot \vec{E}^{inc} ds = \sum_{i=1}^{N_{pw}} c_i^{inc} \int_{\Gamma^{pw}} \vec{W}_j \cdot \vec{W}_i ds$$

- Similarly

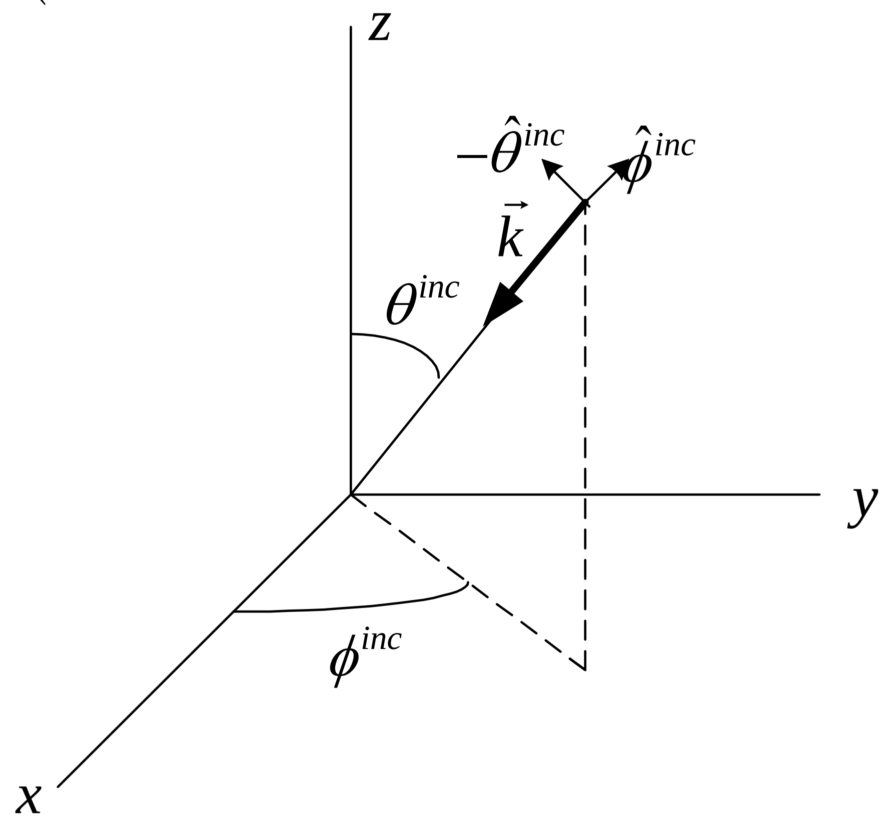
$$\int_{\Gamma^{pw}} \vec{W}_j \cdot \vec{H}^{inc} ds = \sum_{i=1}^{N_{pw}} b_i^{inc} \int_{\Gamma^{pw}} \vec{W}_j \cdot \vec{W}_i ds$$

- Once we solve for c_i^{inc} and b_i^{inc}, these vectors will be multiplied against the local element matrices to fill the right-hand-side of the system matrix.

Plane Wave in a 3D Space

- In spherical coordinates, the plane wave can be said to have an angle of incidence of $(\theta^{inc}, \phi^{inc})$. The plane wave is then propagating along the negative radial direction, with a time-harmonic *k*-vector:

$$\vec{k} = k\left(\sin\theta^{inc}\cos\phi^{inc}\hat{x} + \sin\theta^{inc}\sin\phi^{inc}\hat{y} + \cos\theta^{inc}\hat{z}\right)$$



- The incident field can be described as a superposition of vertical and horizontally polarized waves, which have time-harmonic electric fields:

$$\vec{E}^{inc}_{V-pole}\left(\vec{r};\theta^{inc},\phi^{inc}\right) = E^{V}_{inc}e^{j\vec{k}\cdot\vec{r}}\hat{\theta}^{inc}$$

$$\vec{E}^{inc}_{H-pole}\left(\vec{r};\theta^{inc},\phi^{inc}\right) = E^{H}_{inc}e^{j\vec{k}\cdot\vec{r}}\hat{\phi}^{inc}$$

where,

$$\hat{\theta}^{inc} = \cos\theta^{inc}\cos\phi^{inc}\hat{x} + \cos\theta^{inc}\sin\phi^{inc}\hat{y} - \sin\theta^{inc}\hat{z}$$

$$\hat{\phi} = -\sin\phi^{inc}\hat{x} + \cos\phi^{inc}\hat{y}$$

and

$$\hat{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

- The magnetic field is derived from the electric field via Faraday’s law:

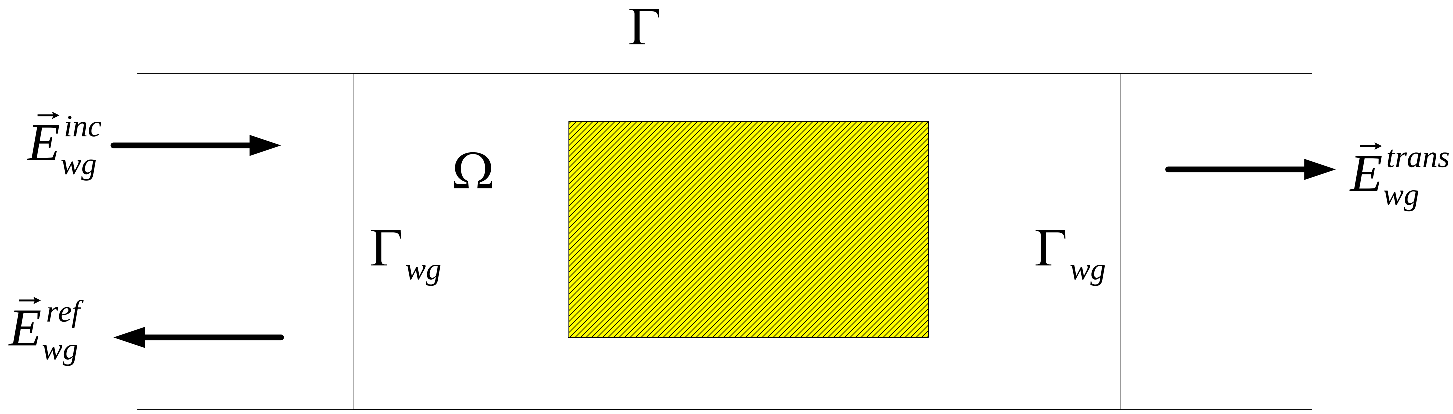
$$\vec{H}^{inc}_{V-pole}\left(\vec{r};\theta^{inc},\phi^{inc}\right) = \frac{-E^{V}_{inc}}{\eta_0}e^{j\vec{k}\cdot\vec{r}}\hat{\phi}^{inc}$$

$$\vec{H}^{inc}_{H-pole}\left(\vec{r};\theta^{inc},\phi^{inc}\right) = \frac{E^{H}_{inc}}{\eta_0}e^{j\vec{k}\cdot\vec{r}}\hat{\theta}^{inc}$$

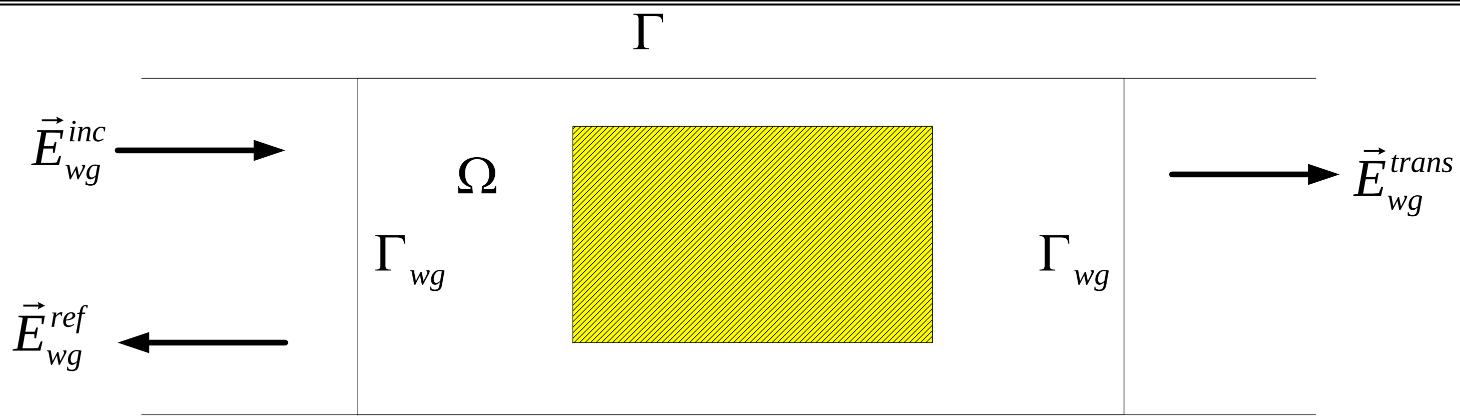
where, η_0 is the characteristic impedance of the host medium (presumably free-space).

Wave-Guide Mode Excitation/Termination

- Consider the FEM modeling of a discontinuity in a waveguide
  - o Ω – the interior domain (FEM) $(\vec{E}, \vec{H})$
  - o Ω_{wg}– the exterior wave guide domain $(\vec{E}_{wg}, \vec{H}_{wg})$

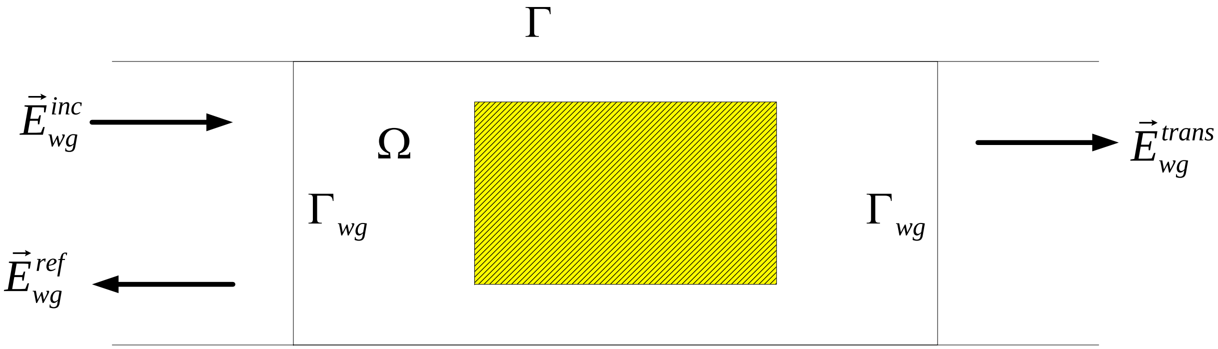


- The domains are separated by the surface Γ_{wg}
  - o Note that the wave-guides do not have to be the same
- If the fields in the exterior domain are known, the can be coupled to the interior fields via the boundary conditions on Γ_{wg}
  - o $\hat{n} \times \vec{E}\Big|_{\Gamma_{wg}^{-}} = \hat{n} \times \vec{E}^{+}\Big|_{\Gamma_{wg}^{+}}$
  - o $\hat{n} \times \vec{H}\Big|_{\Gamma_{wg}^{-}} = \hat{n} \times \vec{H}^{+}\Big|_{\Gamma_{wg}^{+}}$



- In the exterior waveguide region, only the incident fields are assumed to be known
- Objectives:
  - Unique solution for $(\vec{E},\vec{H})$ in Ω and $(\vec{E}_{wg},\vec{H}_{wg})$ in Ω_{wg}.
  - Symmetric Formulation

The Interior FE-Problem



- The interior problem is represented by the standard FEM formulation based on the weak-form of the vector Helmholtz equation:

$$\int_{\Omega}\left(\nabla\times\vec{E}^{a}\cdot\frac{1}{\mu_r}\nabla\times\vec{E}-k_0^2\vec{E}^{a}\cdot\varepsilon_r\vec{E}\right)d\Omega-\int_{\Gamma_{wg}^{-}}\vec{E}^{a}\times\frac{1}{\mu_r}\nabla\times\vec{E}\cdot\hat{n}ds=0$$

- In the boundary integral, we can note that $\because\ \nabla\times\vec{E}=-jk_0\eta_o\mu_r\vec{H}$:

$$-\int_{\Gamma_{wg}^{-}}\vec{E}^{a}\times\frac{1}{\mu_r}\nabla\times\vec{E}\cdot\hat{n}ds=jk_o\eta_o\int_{\Gamma_{wg}^{-}}\vec{E}^{a}\times\vec{H}\cdot\hat{n}ds=-jk_0\eta_o\int_{\Gamma_{wg}^{-}}\vec{E}^{a}\cdot\hat{n}\times\vec{H}ds$$

- Therefore, this can be expressed as:

$$\int_{\Omega}\left(\nabla\times\vec{E}^{a}\cdot\frac{1}{\mu_r}\nabla\times\vec{E}-k_0^2\vec{E}^{a}\cdot\varepsilon_r\vec{E}\right)d\Omega-jk_0\eta_o\int_{\Gamma_{wg}^{-}}\vec{E}^{a}\cdot\hat{n}\times\vec{H}ds=0 \qquad (1)$$

The Exterior Modal-Problem

- Consider the *i*-th wave-guide port. The magnetic field can be expanded into forward and backward traveling modes:

$$\vec{H}_{wg_i} = \vec{H}^{+}_{wg_i} + \vec{H}^{-}_{wg_i}$$

  - where, the forward and backward traveling waves can be expanded via a discrete modal expansion:

$$\vec{H}^{+}_{wg_i} = \sum_{m=1}^{M^+} \vec{H}^{+}_{m} e^{-jk_m l}, \qquad \vec{H}^{-}_{wg_i} = \sum_{m=1}^{M^-} h_m \vec{H}^{-}_{m} e^{+jk_m l}$$

  - where, l is the waveguide axis, k_m is the axial wave-number, and $\vec{H}^{+/-}_{m}$ are the waveguide modes along the cross-section.
  - The incident modes are assumed to be known. The reflected modes are weighted by unknown coefficients h_m
  - The electric fields can be similarly expanded:

$$\vec{E}^{+}_{wg_i} = \sum_{m=1}^{M^+} \vec{E}^{+}_{m} e^{-jk_m l}, \qquad \vec{E}^{-}_{wg_i} = \sum_{m=1}^{M^-} \overline{\overline{Z}}_m \cdot h_m \vec{H}^{-}_{m} e^{+jk_m l}$$

  - $\vec{E}^{+}_{m}$ is again assumed to be known, and $\overline{\overline{Z}}_m$ is a tensor product relating the magnetic and electric fields for each mode.

Exterior Formulation

- On the wave-guide boundary Γ_{wg} of the *i*-th port, we pose:
  - $$\hat{n}\times\vec{H}\Big|_{\Gamma_{wg}^-} = \hat{n}\times\vec{H}^+\Big|_{\Gamma_{wg}^+} = \hat{n}\times\left(\sum_{m=1}^{M^+}\vec{H}_m^+ e^{-jk_m l} + \sum_{m=1}^{M^-} h_m \vec{H}_m^- e^{+jk_m l}\right)\Bigg|_{\Gamma_{wg}^+}$$
  - $$\hat{n}\times\vec{E}\Big|_{\Gamma_{wg}^-} = \hat{n}\times\vec{E}^+\Big|_{\Gamma_{wg}^+} = \hat{n}\times\left(\sum_{m=1}^{M^+}\vec{E}_m^+ e^{-jk_m l} + \sum_{m=1}^{M^-} \overline{\overline{Z}}_m\cdot h_m \vec{H}_m^- e^{+jk_m l}\right)\Bigg|_{\Gamma_{wg}^+}$$
- Inner products are then appropriately performed:
  - $$\int_{\Gamma_{wg}^-}\vec{E}^a\cdot\hat{n}\times\vec{H}ds = \sum_{m=1}^{M^+} e^{-jk_m l_{wg}}\int_{\Gamma_{wg}^+}\vec{E}^a\cdot\hat{n}\times\vec{H}_m^+ ds + \sum_{m=1}^{M^-} h_m e^{+jk_m l_{wg}}\int_{\Gamma_{wg}^+}\vec{E}^a\cdot\hat{n}\times\vec{H}_m^- ds$$
  - $$e^{jk_n l_{wg}}\int_{\Gamma_{wg}^-}\hat{n}\times\vec{H}_n^-\cdot\vec{E}ds = e^{jk_n l_{wg}}\sum_{m=1}^{M^+} e^{-jk_m l_{wg}}\int_{\Gamma_{wg}^+}\hat{n}\times\vec{H}_n^-\cdot\vec{E}_m^+ ds + e^{jk_n l_{wg}}\sum_{m=1}^{M^-} h_m e^{jk_m l_{wg}}\int_{\Gamma_{wg}^+}\hat{n}\times\vec{H}_n^-\cdot\overline{\overline{Z}}_m\cdot\vec{H}_m^- ds$$

Coupled Formulation

- In Ω, away from the waveguide boundary:

$$\int_{\Omega}\left(\nabla\times\vec{E}^a\cdot\frac{1}{\mu_r}\nabla\times\vec{E}-k_0^2\vec{E}^a\cdot\varepsilon_r\vec{E}\right)d\Omega=0$$

- In Ω, near the waveguide boundary:

$$\int_{\Omega}\left(\nabla\times\vec{E}^a\cdot\frac{1}{\mu_r}\nabla\times\vec{E}-k_0^2\vec{E}^a\cdot\varepsilon_r\vec{E}\right)d\Omega+$$

$$-jk_0\eta_o\sum_{m=1}^{M^-}h_m e^{+jk_m l_{wg}}\int_{\Gamma_{wg}^+}\vec{E}^a\cdot\hat{n}\times\vec{H}_m^- ds=jk_0\eta_o\sum_{m=1}^{M^+}e^{-jk_m l_{wg}}\int_{\Gamma_{wg}^+}\vec{E}^a\cdot\hat{n}\times\vec{H}_m^+ ds$$

- On the waveguide boundary:

$$-jk_o\eta_o e^{jk_n l_{wg}}\int_{\Gamma_{wg}^-}\hat{n}\times\vec{H}_n^-\cdot\vec{E}ds+jk_o\eta_o e^{jk_n l_{wg}}\sum_{m=1}^{M^-}h_m e^{jk_m l_{wg}}\int_{\Gamma_{wg}^+}\hat{n}\times\vec{H}_n^-\cdot\overline{\overline{Z}}_m\cdot\vec{H}_m^- ds$$

$$=-jk_o\eta_o e^{jk_n l_{wg}}\sum_{m=1}^{M^+}e^{-jk_m l_{wg}}\int_{\Gamma_{wg}^+}\hat{n}\times\vec{H}_n^-\cdot\vec{E}_m^+ ds$$

Discretization

- In Ω, the electric field is expanded via H(p)-curl-conforming basis functions on three-dimensional elements
- The test functions in Ω are also expanded with the same basis function space
- On Γ_{wg}, the electric field is also expressed via the H(p)-curl conforming basis representative of the face and edge functions of the 3D cell that lie on Γ_{wg}
- The modal fields $\vec{H}_m^-$ are known (exactly or approximately), and have full support over the entire Γ_{wg} surface.
- System Matrix:

$$\begin{pmatrix} \overline{\overline{K}}_{\Omega,\Omega} & \overline{\overline{K}}_{\Omega,\Gamma} & 0 \\ \overline{\overline{K}}_{\Omega,\Gamma}^T & \overline{\overline{K}}_{\Gamma,\Gamma} & \overline{\overline{P}} \\ 0 & \overline{\overline{P}}^T & \overline{\overline{Z}} \end{pmatrix} \begin{pmatrix} \overline{e}_\Omega \\ \overline{e}_\Gamma \\ \overline{h} \end{pmatrix} = \begin{pmatrix} 0 \\ b_h \\ b_e \end{pmatrix}$$

- where,

$$\left(K_{\Omega,\Omega}\right)_{i,j} = \int_\Omega \left(\nabla \times \vec{w}_i \cdot \frac{1}{\mu_r} \nabla \times \vec{w}_j - k_0^2 \vec{w}_i \cdot \varepsilon_r \vec{w}_j \right) d\Omega$$

and $K_{\Omega,\Gamma}$ is similarly defined with $\vec{w}_j$ supported on Γ. (similar for $K_{\Gamma,\Gamma}$)

- The coupling matrix block is:

$$\left(P\right)_{i,m} = -jk_0\eta_o e^{+jk_m l_{wg}} \int_{\Gamma_{wg}^+} \vec{w}_j \cdot \hat{n}\times\vec{H}_m^- ds$$

- The diagonal block based on the waveguide modes is:

$$\left(Z\right)_{n,m} = jk_o\eta_o e^{jk_n l_{wg}} e^{jk_m l_{wg}} \int_{\Gamma_{wg}^+} \hat{n}\times\vec{H}_n^- \cdot \bar{\bar{Z}}_m \cdot \vec{H}_m^- ds$$

- The right-hand side vectors are computed as:

$$\left(b_m\right)_j = jk_0\eta_o \sum_{m=1}^{M^+} e^{-jk_m l_{wg}} \int_{\Gamma_{wg}^+} \vec{w}_j \cdot \hat{n}\times\vec{H}_m^+ ds$$

$$\left(b_e\right)_n = -jk_o\eta_o e^{jk_n l_{wg}} \sum_{m=1}^{M^+} e^{-jk_m l_{wg}} \int_{\Gamma_{wg}^+} \hat{n}\times\vec{H}_n^- \cdot \vec{E}_m^+ ds$$

- Note that System Matrix is symmetric providing that the block matrix $\bar{\bar{Z}}$ is symmetric.
  - Note, in general, due to orthogonality, $\bar{\bar{Z}}$ should be a diagonal matrix.