Basis Function Spaces for 3D FEM

- Three-dimensional FEM solution methods for dynamic Electromagnetic simulations must be discretized via curl-conforming vector function spaces
  - Enforces tangential continuity of fields
  - Allows normal discontinuity of fields (step discontinuity in media constant)
  - Can weakly constrain normal flux density
  - Properly models Gauss' law in weak sense
    - Eliminates spurious solutions
- The Whitney element is an H(0)-curl conforming basis
  - Constant tangent projection on any shared boundary
  - Linear normal
  - Constant Curl (complete to zero-th order)
- H(0)-curl basis is low-order convergent (~1st order in many cases)
- It is highly desirable to build higher-order basis function spaces to improve rates of convergence of solution
  - Cost: additional degrees of freedom per element
  - Gain: fewer degrees of freedom globally to realize a desired solution accuracy.

High-Order Basis Function Spaces for 3D FEM

- There have been two philosophical paths for developing high-order basis function spaces
    - Hierarchical Basis Function Spaces[1]
        - Topologically based basis function space
    - Interpolatory Basis Function Spaces[2]
        - Cell based basis function space
        - Interpolatory – at distinct points on patch, one function has unit amplitude, the remaining functions are zero at that point, or have zero tangential projection

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[1] J. P. Webb, "Hierarchical vector basis function spaces of arbitrary order for triangular and tetrahedral finite elements," IEEE Transactions on Antennas and Propagation, Vol. 47, No. 8, pp. 1244-1253, August 1999.

[2] R. Graglia, D. Wilton, and A. F. Peterson, "Higher Order Interpolatory Vector Bases for Computational Electromagnetics," IEEE Transactions on Antennas and Propagation, Vol. 45, No. 3, pp. 329-342, March 1997.

Hierarchical Basis function Spaces

- Advantages:
  - Topological association of each basis function space
    - Can easily incorporate local *p*-refinement
      - Order is associated with sub-topology (edge/face/volume), not the entire cell
  - Low-frequency modeling (or fine-discretizations)
    - At low frequencies, there is a natural decoupling of the rotational (dynamic) and irrotational (static) spaces.
    - Hierarchical basis proposed naturally decouple these two spaces
      - Explicit for all order basis > H(0)
      - The H(0) basis must be decomposed further using Tree/Co-tree type decompositions
  - Mixed-Order and Polynomial Complete basis can be combined in the same simulation space
- Disadvantages:
  - High condition number as *p* gets modestly large
    - Remedy – can attempt to orthogonalize the basis function spaces

Interpolatory Basis function Spaces

- Advantages:
  - Fairly simple to implement
  - Well conditioned through moderately large orders of p
- Disadvantages
  - Can't locally p-refine (at least not easily)
    - Global basis order
  - Prone to low-frequency breakdown
    - The rotational and irrotational function spaces are not distinct
    - Loop-tree type decompositions must be applied to all basis orders
  - Cannot easily combine mixed order and polynomial complete order basis function spaces.

“To Complete or Not To Complete?”

- When building a vector function space to order *p*, the question arises as to whether one should build a polynomial space that is complete to order *p* or use a mixed order.
  - When referring to “complete”, this would imply that basis functions associated with each independent base vector are complete in all dimensions.
  - When referring to “mixed-order”, the basis functions are complete to order *p* in all dimensions, but are incomplete to order *p+1* along some dimensions relative to the base vector.
    - For curl conforming elements, the spatial variation is incomplete to orders *p+1* relative to axes that have non-zero orthogonal projections relative to the base vector.
    - The H(0)-curl basis is a mixed-order basis. Each edge function has a constant tangential projection, and a linear normal projection.
    - For curl conforming elements, the curl is complete to order *p*.
- It has been found that when modeling smooth boundaries without material discontinuities, complete polynomial spaces are optimal
- When discontinuities and singular geometries are present, mixed-order are necessary.

Hierarchical Basis function Spaces

- The Hierarchical basis function space discussed here is that presented by J. P. Webb, in his 99 IEEE T-AP paper.
- The basis function space is based on a decomposition of the function space into pure gradient functions (which are irrotational), and functions representing the rotational space.
  - Advantages of the decomposition:
    - The function space can be used to construct complete or incomplete (mixed-order) basis, or a combination.
    - The function space can readily be used for low-frequency formulations where low-frequency breakdown of the Helmholtz operator is a concern.
- The basis functions are based on topological function spaces:
  - Edge space
  - Face space
  - Volume space
- Topological function spaces can be useful if local *p*-refinement of the basis is desired.

Hierarchical Basis functions for the Simplex

- The Hierarchical basis function space are given here for the simplex.
    - 1D Simplex – edge function space
    - 2D Simplex – face function space
    - 3D Simplex – volume function space
- The function space can thus be applied to 1D, 2D, and 3D simplices.
    - Note that the *N*D simplex inherits all the function spaces of the lower degrees.
- The initial view of the function spaces appear to be complex. In reality they are quite simple in form.
- In general, we will discuss the function space in the local coordinates of the simplex (e.g., 1D simplex (L_0, L_1), 2D simplex (L_0, L_1, L_2), 3D simplex (L_0, L_1, L_2, L_3)).
    - Note that when 1D and 2D simplex functions are inherited by the 3D simplex, they are defined by their local coordinates that span the entire 3D simplex.

Edge Function Space

- Each edge of a simplex is assigned an *edge function space*. The function space is defined by the two edge coordinates of the edge. However, the function spans the entire simplex.
- Consider an edge bound by nodes 0, 1. Thus, the edge is defined by local coordinates L_0, L_1.
    - Note that this can be generalized to an edge bound by nodes *i*, *j*, and thus the edge is defined by local coordinates L_i, L_j.
- For higher-order simplices (2D or 3D), each edge is assigned an edge function space.
- The function space:
    - Enforces tangential continuity across the edge to all simplices that own the edge.
    - For 3D simplices (tetrahedron), enforces tangential continuity across any face that owns the edge
    - The function space spans all simplices owning the edge, and is described by the two coordinates of the edge (e.g., L_i, L_j) throughout the simplex

The Edge Rotational Space

- Rotational Space and the Whitney Element:
    - The rotational space of the edge function space only consists of the H(0)-curl basis function, which is also referred to as the Whitney basis:

$$\vec{W}_{0,1} = L_0 \nabla L_1 - L_1 \nabla L_0$$

    - The curl of the Whitney basis is evaluated in closed form:

$$\nabla \times \vec{W}_{0,1} = 2 \nabla L_0 \times \nabla L_1$$

- Observation:
    - It is noted that the Whitney element actually comprises *both* the 0-th order gradient space, *and* the 1st-order rotational space.
        - This will be more apparent when we discuss low-frequency break down issues and decompositions of this function space.

Edge Function Space – The Gradient Space

- The H(*p*)-curl function space for *p* > 0 are pure gradient basis:

$$\vec{G}_p^{(e)} = \nabla Q_{p-1}\left(L_0, L_1\right), \qquad p \geq 1$$

where

$$Q_p\left(L_0, L_1\right) = L_0 L_1 \left(L_0 - L_1\right)^p$$

- Note that $Q_{p-1}\left(L_1, L_2\right)$ is order *p* in both L_0 and L_1 & $Q_{p-1}\left(L_1, L_2\right)$ = 0 at the ends of the edge
- Expanding the gradient:

$$\vec{G}_p^{(e)} = \nabla L_0 L_1 \left(L_0 - L_1\right)^{p-1}$$

$$= \left(L_0 - L_1\right)^{p-2}\left[L_1\left(pL_0 - L_1\right)\nabla L_0 + L_0\left(L_0 - pL_1\right)\nabla L_1\right]$$

  - Note for *p* = 1:
    - $\vec{G}_1^{(e)} = \nabla L_0 L_1 = L_1 \nabla L_0 + L_0 \nabla L_1$
- Curl:
  - $\nabla \times \vec{G}_p^{(e)} = 0$
- There are a total of *p* + 1 edge basis for the H(*p*)-curl function space.

Face Function Spaces

- Each face of the simplex is assumed a function space.
- The face is defined by vertices (0,1,2), and thus described by local coords (L_0, L_1, L_2)
    - Note that this can be generalized to a face bound by vertices (*i*,*j*,*k*)
- For a 2D simplex (triangle):
    - the face functions are internal to the face only, and have zero in-plane tangential projection on any edge.
- For a 3D simplex (tetrahedron):
    - Each face is assigned a function space. The functions span the entire tetrahedron that owns the face.
    - The face functions have zero tangential projection on any edge of the tetrahedron, and zero tangential projection on all other faces.
    - The face function space enforces tangential continuity across its *face* shared between two tetrahedron that own the face.

- The root function for a face function space up to order *p* (for *p* > 0) is defined as:

$$F_{\varphi,i} = \Psi^{(i)} Q_{\varphi-3i-2}\left(L_1, L_2\right); \qquad i \le \tau - 1 \quad \left(3 \text{ functions}\right)$$

  and

$$F_{\varphi,\tau} = \begin{cases} \Psi^{(\tau)}, & \text{if } \left(\varphi \,\%\, 3\right) = 0 \quad \left(1 \text{ function}\right) \\ \Psi^{(\tau)}\left(L_1 - L_2\right), & \text{if } \left(\varphi \,\%\, 3\right) = 1 \quad (2 \text{ functions}) \\ \Psi^{(\tau)} L_1 L_2, & \text{if } \left(\varphi \,\%\, 3\right) = 2 \quad (3 \text{ functions}) \end{cases}$$

  - where

$$\tau = \operatorname{int}\left(\varphi / 3\right)$$

  - and, $\Psi^{(i)}$ is a “Bubble function”, defined as:

$$\Psi^{(i)} = \left(L_0 L_1 L_2\right)^i$$

    - (Note that the bubble function is zero along all outer boundaries)
  - Note that the multiple “functions” are defined by permuting the local coordinate indices from the base coordinate value.

Face Gradient Space

- The Face Gradient Space is defined as:

$$\vec{G}^{(f)}_{p,j} = \nabla F_{p+1,j+1}, \quad \text{for } j = 0,1,\ldots,\text{int}\left(\frac{p-1}{3}\right)-1$$

where, $F_{p+1,j+1}$ is our root function defined on page 12, and $\varphi = p+1$.

- There will be a total of $p(p-1)/2$ gradient face functions for the *p*-th order face gradient space.
  - When increasing the order from order *p-1* to order *p*, there will an additional *p-1* face gradient functions that are needed.
- The Face Gradient Space by definition has zero curl.
- The lowest order face gradient function space is for ***p* = 2**:
  - $\varphi = 3,\ \tau = 1, (\varphi\%3) = 0$; thus there will only be one function:

$$\vec{G}^{(f)}_{2,0} = \nabla F_{3,1} = \nabla\Psi^{(1)} = \vec{G}^{(f)}_{2,0} = \nabla\Psi^{(1)}$$

$$= L_1 L_2 \nabla L_0 + L_0 L_2 \nabla L_1 + L_0 L_1 \nabla L_2$$

- The next face gradient function space is ***p* = 3**. This inherits the functions from *p* = 2, and adds an additional 2 functions:
  - o $\varphi = 4,\ \tau = 1, \left(\varphi \% 3\right) = 1$; thus there will only be two functions:

$$\vec{G}_{3,0}^{(f)} = \nabla F_{4,1} = \nabla\left(\Psi^{(1)}\left(L_1 - L_2\right)\right)$$

Therefore,

$$\vec{G}_{3,0}^{(f)} = \left(L_1 - L_2\right)\nabla\Psi^{(1)} + \Psi^{(1)}\left(\nabla L_1 - \nabla L_2\right)$$

$$\Re\left(\vec{G}_{3,0}^{(f)}\right) = \left(L_2 - L_0\right)\nabla\Psi^{(1)} + \Psi^{(1)}\left(\nabla L_2 - \nabla L_0\right)$$

where, $\nabla\Psi^{(1)} = \vec{G}_{2,0}^{(f)}$

- The next face gradient function space is ***p* = 4**. This inherits the functions from *p* = 3, and adds an additional 3 functions:
  - o $\varphi = 5,\ \tau = 1, \left(\varphi \% 3\right) = 2$; thus there will be three functions for $\vec{G}_{4,0}^{(f)} = \nabla F_{5,1}$:

$$\vec{G}_{4,0}^{(f)} = \nabla\left(\Psi^{(1)} L_1 L_2\right) = L_1 L_2 \nabla\Psi^{(1)} + \Psi^{(1)}\left(L_2 \nabla L_1 + L_1 \nabla L_2\right)$$

$$\Re\left(\vec{G}_{4,0}^{(f)}\right) = L_2 L_0 \nabla\Psi^{(1)} + \Psi^{(1)}\left(L_0 \nabla L_2 + L_2 \nabla L_0\right)$$

$$\Re\Re\left(\vec{G}_{4,0}^{(f)}\right) = L_0 L_1 \nabla\Psi^{(1)} + \Psi^{(1)}\left(L_1 \nabla L_0 + L_0 \nabla L_1\right)$$

- The next face gradient function space is ***p* = 5**. This inherits the functions from *p* = 4, and adds an additional 4 functions:
    - $\varphi = 6,\ \tau = 2, (\varphi\%3) = 0$; thus there will be three functions for $\vec{G}^{(f)}_{5,0} = \nabla F_{6,1}$, and one function for $\vec{G}^{(f)}_{5,1} = \nabla F_{6,2}$:

$$\vec{G}^{(f)}_{5,0} = \nabla\left(\Psi^{(1)} Q_1\left(L_1, L_2\right)\right) = Q_1\left(L_1, L_2\right)\nabla\Psi^{(1)} + \Psi^{(1)}\nabla Q_1\left(L_1, L_2\right)$$

$$\Re\left(\vec{G}^{(f)}_{5,0}\right) = \nabla\left(\Psi^{(1)} Q_1\left(L_2, L_0\right)\right) = Q_1\left(L_2, L_0\right)\nabla\Psi^{(1)} + \Psi^{(1)}\nabla Q_1\left(L_2, L_0\right)$$

$$\Re\Re\left(\vec{G}^{(f)}_{5,0}\right) = \nabla\left(\Psi^{(1)} Q_1\left(L_0, L_1\right)\right) = Q_1\left(L_0, L_1\right)\nabla\Psi^{(1)} + \Psi^{(1)}\nabla Q_1\left(L_0, L_1\right)$$

$$\vec{G}^{(f)}_{5,1} = \nabla\Psi^{(2)}$$

where

$$\nabla\Psi^{(i)} = \nabla\left(L_0 L_1 L_2\right)^i = i\left(L_o L_1 L_2\right)^{i-1}\left(L_1 L_2 \nabla L_0 + L_0 L_2 \nabla L_1 + L_0 L_1 \nabla L_2\right)$$

- We can continue to build higher-order gradient space functions in a similar manner.

Face Rotational Space

- The Face Rotational Space is defined as:

$$\vec{R}^{(f)}_{p,j} = \left(L_2\nabla L_0 - L_0\nabla L_2\right)F_{p-1,j}; \qquad j = 0,1,\ldots,\sigma-1 \ \left(3\text{ functions}\right)$$

  - where, $\sigma = \text{int}\left(\left(p-1\right)/3\right)$ and $F_{p-1,j}$ is the root function defined on page 12.
  - For $j=\sigma$:

$$\vec{R}^{(f)}_{p,\sigma} = \begin{cases} \left(L_2\nabla L_0 - L_0\nabla L_2\right)F_{p-1,\sigma}, & \text{if}\left(p\ \%\ 3\right)=0\ \left(3\text{ functions}\right) \\ \begin{bmatrix} \left(L_2-L_1\right)\nabla L_0 + \left(L_0-L_2\right)\nabla L_1 \\ +\left(L_1-L_0\right)\nabla L_2 \end{bmatrix}\Psi^{(\sigma)} & \text{if}\left(p\ \%\ 3\right)=1\left(1\text{ function}\right) \\ \left[L_0\left(L_1\nabla L_2 - L_2\nabla L_1\right)\right]\Psi^{(\sigma)} & \text{if}\left(p\ \%\ 3\right)=2\left(2\text{ functions}\right) \end{cases}$$

  - Note that the above expressions can be expressed in terms of the Whitney edge functions! (next page)

$$\sigma = \text{int}\left(\left(p-1\right)/3\right),$$

$$\vec{R}_{p,j}^{(f)} = \vec{W}_{2,0} F_{p-1,j}; \qquad j = 0,1,\ldots,\sigma-1 \ \left(3 \text{ functions}\right)$$

- For $j = \sigma$:

$$\vec{R}_{p,\sigma}^{(f)} = \begin{cases} F_{p-1,\sigma}\vec{W}_{2,0}, & \text{if } \left(p \ \% \ 3\right) = 0 \ \left(3 \text{ functions}\right) \\ \Psi^{(\sigma)}\left[\vec{W}_{1,2} + \vec{W}_{2,0} + \vec{W}_{0,1}\right], & \text{if } \left(p \ \% \ 3\right) = 1 \left(1 \text{ function}\right) \\ \Psi^{(\sigma)} L_0 \vec{W}_{1,2}, & \text{if } \left(p \ \% \ 3\right) = 2 \left(2 \text{ functions}\right) \end{cases}$$

- where,

$$\vec{W}_{i,j} = L_i \nabla L_j - L_j \nabla L_i$$

- Next, we need to compute the curl of the basis. This is aided by the identity:

$$\nabla \times \left(\phi \nabla L_i\right) = \nabla \phi \times \nabla L_i$$

- Furthermore:

$$\nabla \times \left[F\left(L_1 \nabla L_2 - L_2 \nabla L_1\right)\right] = \nabla F \times \left(L_1 \nabla L_2 - L_2 \nabla L_1\right) + F\left(\nabla L_1 \times \nabla L_2 - \nabla L_2 \times \nabla L_1\right)$$

$$= \nabla F \times \left(L_1 \nabla L_2 - L_2 \nabla L_1\right) + 2F \nabla L_1 \times \nabla L_2$$

Curl of the Face Rotational Space

- The Curl of the Face Rotational Space is defined as:

$$\nabla\times\vec{R}_{p,j}^{(f)} = \nabla F_{p-1,j}\times\vec{W}_{2,0} + 2F_{p-1,j}\vec{a}_{2,0}; \quad j=0,1,\ldots,\sigma-1 \ (\text{3 functions})$$

$$\nabla\times\vec{R}_{p,\sigma}^{(f)} = \begin{cases} \nabla F_{p-1,\sigma}\times\vec{W}_{2,0} + 2F_{p-1,\sigma}\vec{a}_{2,0}, & \text{if } (p\ \%\ 3)=0\ (\text{3 functions}) \\ \nabla\Psi^{(\sigma)}\times\left[\vec{W}_{1,2}+\vec{W}_{2,0}+\vec{W}_{0,1}\right] + 2\Psi^{(\sigma)}\left[\vec{a}_{1,2}+\vec{a}_{2,0}+\vec{a}_{0,1}\right], & \text{if } (p\ \%\ 3)=1\ (\text{1 function}) \\ \nabla\Psi^{(\sigma)}\times\left[L_0\vec{W}_{1,2}\right] + \Psi^{(\sigma)}\left[2L_0\vec{a}_{1,2} - L_1\vec{a}_{2,0} - L_2\vec{a}_{0,1}\right], & \text{if } (p\ \%\ 3)=2\ (\text{2 functions}) \end{cases}$$

  - where

$$\vec{a}_{i,j} = \nabla L_i \times \nabla L_j$$

Building the Face Rotational Space

- The lowest order face rotational space occurs for *p* = 2.
  - $\sigma = \text{int}\left(\dfrac{p-1}{3}\right) = 0$, and (*p* % 3) = 2. Therefore,

$$\vec{R}_{2,0}^{(f)} = \Psi^{(0)}\left[L_0\vec{W}_{1,2}\right] = L_0\vec{W}_{1,2}$$

$$\Re\left(\vec{R}_{2,0}^{(f)}\right) = L_1\vec{W}_{2,0}$$

  - where, $\Re$ represents the permutation operator.
  - Similarly:

$$\nabla\times\vec{R}_{2,0}^{(f)} = 2L_0\vec{a}_{1,2} - L_1\vec{a}_{2,0} - L_2\vec{a}_{0,1}$$

$$\Re\left(\nabla\times\vec{R}_{2,0}^{(f)}\right) = 2L_1\vec{a}_{2,0} - L_2\vec{a}_{0,1} - L_0\vec{a}_{1,2}$$

- The next order rotational space is $p = 3$. The 3rd-order space inherits the 2nd-order functions, plus three additional functions, defined by the following:
  - $\sigma = \mathrm{int}\left(\frac{p-1}{3}\right) = 0$, and ($p$ % 3) = 0. Therefore,

$$\vec{R}_{3,0}^{(f)} = L_1 L_2 \vec{W}_{2,0}$$

$$\Re\left(\vec{R}_{3,0}^{(f)}\right) = L_2 L_0 \vec{W}_{0,1}$$

$$\Re\Re\left(\vec{R}_{3,0}^{(f)}\right) = L_0 L_1 \vec{W}_{1,2}$$

- Similarly, the curl is defined as:

$$\nabla \times \vec{R}_{3,0}^{(f)} = \left(L_1 \nabla L_2 + L_2 \nabla L_1\right) \times \vec{W}_{2,0} + 2 L_1 L_2 \vec{a}_{2,0}$$

$$\Re\left(\nabla \times \vec{R}_{3,0}^{(f)}\right) = \left(L_2 \nabla L_0 + L_0 \nabla L_2\right) \times \vec{W}_{0,1} + 2 L_2 L_0 \vec{a}_{0,1}$$

$$\Re\Re\left(\nabla \times \vec{R}_{3,0}^{(f)}\right) = \left(L_0 \nabla L_1 + L_1 \nabla L_0\right) \times \vec{W}_{1,2} + 2 L_0 L_1 \vec{a}_{1,2}$$

- The next order rotational space is *p* = 4. The 4rd-order space inherits the 3rd-order functions, plus four additional functions, defined by the following:
  - o $\sigma = \text{int}\left(\frac{p-1}{3}\right) = 1$, and (*p* % 3) = 1. Therefore,

$$\vec{R}_{4,0}^{(f)} = L_1 L_2 \left(L_1 - L_2\right)\vec{W}_{2,0}$$

$$\Re\left(\vec{R}_{4,0}^{(f)}\right) = L_2 L_0 \left(L_2 - L_0\right)\vec{W}_{0,1}$$

$$\Re\Re\left(\vec{R}_{4,0}^{(f)}\right) = L_0 L_1 \left(L_0 - L_1\right)\vec{W}_{1,2}$$

$$\vec{R}_{4,1}^{(f)} = \Psi^{(1)}\left[\vec{W}_{1,2} + \vec{W}_{2,0} + \vec{W}_{0,1}\right]$$

- The curl can easily be derived for these functions

- Note that we can continue building basis functions to higher order in a similar manner.
  - o There are a total of $(p-1)(p+2)/2$ rotational basis for the *p*-th order face rotational space.
  - o When increasing the order from *p-1* to *p*, there will be an additional *p* face rotational functions needed.

The Volume Function Space

- The volume basis function space is internal to the volume only
- The basis function space has the properties that it completes the 3D simplex function space to the desired order
- The vector volume functions also have zero tangential projection on any edge or face topology.
  - Either purely normal to the face (gradient), or is zero on the face.

The Volume Gradient Function Space

- The volume gradient function space is again derived via perfect gradients.
- There are a total of $p\left(p-1\right)\left(p-2\right)/6$ volume gradient basis for the *p*-th order function space ($p \geq 3$).
- The number of additional functions added from the volume gradient space from order *p-1* to order *p* is $\left(p-2\right)\left(p-3\right)/2$ functions.
- The basis used for the gradient space is:

$$V_{p,i,j} = (L_0 L_1 L_2 L_3) L_1^{p-4-i} L_2^{i-j} L_3^{j}, \quad (i = 0..p-4;\ j = 0..i)$$

- The gradient space is defined as:

$$\vec{G}^{(v)}_{p,i,j} = \nabla V_{p+1,i,j}, \qquad \left(i = 0..p-3;\quad j = 0..i;\ p \geq 3\right)$$

- Thus:

$$\begin{aligned}\vec{G}^{(v)}_{p,i,j} = {} & L_1^{p-2-i} L_2^{i-j+1} L_3^{j+1} \nabla L_0 + \left(p-2-i\right) L_0 L_1^{p-3-i} L_2^{i-j+1} L_3^{j+1} \nabla L_1 + \\ & \left(i-j+1\right) L_0 L_1^{p-2-i} L_2^{i-j} L_3^{j+1} \nabla L_2 + \left(j+1\right) L_0 L_1^{p-2-i} L_2^{i-j+1} L_3^{j} \nabla L_3\end{aligned}$$

$$\left(i = 0..p-3;\quad j = 0..i;\ p \geq 3\right)$$

Examples of the Gradient Function Space

- The lowest order gradient volume basis function is $p = 3$.
  - There is only 1 gradient volume basis for $p = 3$:

$$\vec{G}^{(v)}_{3,0,0} = \nabla\left(L_0L_1L_2L_3\right) = L_0L_2L_3\nabla L_1 + L_0L_1L_3\nabla L_2 + L_0L_1L_2\nabla L_3 + L_1L_2L_3\nabla L_0$$

- For $p = 4$, there are 3 additional functions added:

$$\begin{aligned}
\vec{G}^{(v)}_{4,0,0} &= \nabla L_0L_1^2L_2L_3 \\
&= L_1^2L_2L_3\nabla L_0 + 2L_0L_1L_2L_3\nabla L_1 + L_0L_1^2L_3\nabla L_2 + L_0L_1^2L_2\nabla L_3 \\
\vec{G}^{(v)}_{4,1,0} &= \nabla L_0L_1L_2^2L_3 \\
&= L_1L_2^2L_3\nabla L_0 + L_0L_2^2L_3\nabla L_1 + 2L_0L_1L_2L_3\nabla L_2 + L_0L_1L_2^2\nabla L_3 \\
\vec{G}^{(v)}_{4,1,1} &= \nabla L_0L_1L_2L_3^2 \\
&= L_1L_2L_3^2\nabla L_0 + L_0L_2L_3^2\nabla L_1 + L_0L_1L_3^2\nabla L_2 + 2L_0L_1L_2L_3\nabla L_3
\end{aligned}$$

- For $p = 5$, there are 6 additional functions added, etc.

The Volume Rotational Function Space

- The volume rotational space has dimension $(p-2)(p-1)(2p+3)/6$, such that when the order increases from order *p-1* to order *p*, an additional $p(p-2)$ rotational basis functions must be introduced.
- The function space is defined as:

$$i = 0..p-3;$$

$$\vec{R}^{(v)}_{p,i,j} = \left(L_1 L_2 L_3 \nabla L_0\right) L_1^{p-3-i} L_2^{i-j} L_3^{j}, \qquad \left(j = 0..i\right)$$

$$\vec{R}^{(v)}_{p,i,i+1+j} = (L_2 L_3 L_0 \nabla L_1) L_2^{p-3-i} L_3^{i-j} L_0^{j}, \quad \left(j = 0..i\right)$$

$$\vec{R}^{(v)}_{p,i,2i+2} = (L_3 L_0 L_1 \nabla L_2) L_3^{p-3-i} L_0^{i}$$

- The curl of the rotational space functions is defined as:

$$i = 0..p-3;$$

$$\nabla \times \vec{R}^{(v)}_{p,i,j} = \nabla\left(L_1 L_2 L_3 L_1^{p-3-i} L_2^{i-j} L_3^{j}\right) \times \nabla L_0, \qquad \left(j = 0..i\right)$$

$$\nabla \times \vec{R}^{(v)}_{p,i,i+1+j} = \nabla(L_2 L_3 L_0 L_2^{p-3-i} L_3^{i-j} L_0^{j}) \times \nabla L_1, \quad \left(j = 0..i\right)$$

$$\nabla \times \vec{R}^{(v)}_{p,i,2i+2} = \nabla(L_3 L_0 L_1 L_3^{p-3-i} L_0^{i}) \times \nabla L_2$$

Examples of The Volume Rotational Function Space

- The lowest order rotational volume basis is p = 3, which has 3 functions:

$$\vec{R}^{(v)}_{3,0,0} = \left(L_1L_2L_3\nabla L_0\right)L_1^0L_2^0L_3^0 = L_1L_2L_3\nabla L_0$$

$$\vec{R}^{(v)}_{3,0,1} = \left(L_2L_3L_0\nabla L_1\right)L_2^0L_3^0L_0^0 = L_2L_3L_0\nabla L_1$$

$$\vec{R}^{(v)}_{3,0,2} = \left(L_3L_0L_1\nabla L_2\right)L_3^0L_0^0 = L_3L_0L_1\nabla L_2$$

- For p = 3, 8 additional functions are added:

$$\vec{R}^{(v)}_{4,0,0} = \left(L_1L_2L_3\nabla L_0\right)L_1^1L_2^0L_3^0 = L_1^2L_2L_3\nabla L_0$$

$$\vec{R}^{(v)}_{4,0,1} = (L_2L_3L_0\nabla L_1)L_2^1L_3^0L_0^0 = L_2^2L_3L_0\nabla L_1$$

$$\vec{R}^{(v)}_{4,0,2} = (L_3L_0L_1\nabla L_2)L_3^1L_0^0 = L_0L_1L_3^2\nabla L_2$$

$$\vec{R}^{(v)}_{4,1,0} = \left(L_1L_2L_3\nabla L_0\right)L_1^0L_2^1L_3^0 = L_1L_2^2L_3\nabla L_0$$

$$\vec{R}^{(v)}_{4,1,2} = (L_2L_3L_0\nabla L_1)L_2^0L_3^1L_0^0 = L_0L_2L_3^2\nabla L_1$$

$$\vec{R}^{(v)}_{4,1,4} = (L_3L_0L_1\nabla L_2)L_3^0L_0^1 = L_0^2L_1L_3\nabla L_2$$

$$\vec{R}^{(v)}_{4,1,1} = \left(L_1L_2L_3\nabla L_0\right)L_2^0L_3^0L_0^1 = L_1L_2L_3^2\nabla L_0$$

$$\vec{R}^{(v)}_{4,1,3} = (L_2L_3L_0\nabla L_1)L_2^0L_3^0L_0^1 = L_0^2L_2L_3\nabla L_1$$

Building a Hierarchical Basis Function Space

- Now that the function spaces for each topology has been introduced, we can proceed to construct the function space for the entire simplex. We will discuss the tetrahedron, which will consist of a superposition of edge, face, and volume function spaces.
    - Note that a triangle will support face and edge functions
    - A 1D mesh, would support edge functions only
- The hierarchical space can be used to construct mixed-order and polynomial order function spaces.
- Mixed-Order Function space: H(p)-curl conforming basis
    - Complete the rotational space to order p+1 on all sub-topologies
    - Complete the gradient space to order p on all sub-topologies
        - Note the lowest order mixed-order function space is H(0)
- Polynomial Complete Function spaces:
    - Complete the rotational space to order p on all sub-topologies
    - Complete the gradient space to order p on all sub-topologies
        - The lowest order polynomial complete function space is 1st order.

H(*p*)-Curl Function Space (Mixed-Order) on a Tetrahedron

- The H(0)-curl function space ($p = 0$) consists of:
  - Edge space (contains both 0th order gradient edge space, and the 1st-order rotational edge space)
    - 1 basis for each of the 6 edges
- The H(1)-curl function space ($p = 1$) consists of:
  - Edge Space (2 per edge): Ex: Edge (i,j):

$$\vec{W}_{i,j} = L_i \nabla L_j - L_i \nabla L_j$$

$$\vec{G}_1^{(e)} = L_i \nabla L_j + L_j \nabla L_i$$

  - Face Space (2 per face): Ex: Face (i,j,k):

$$\vec{R}_{2,0}^{(f)} = L_i \vec{W}_{j,k}$$

$$\Re\left(\vec{R}_{2,0}^{(f)}\right) = L_j \vec{W}_{k,i}$$

  - Volume (0 functions)

H(*p*)-Curl Function Space (Mixed-Order) on a Tetrahedron

- The H(2)-curl function space ($p = 2$) consists of:
  - Edge Space (3 per edge): Ex: Edge (i,j):

$$\vec{W}_{i,j} = L_i \nabla L_j - L_i \nabla L_j$$

$$\vec{G}_1^{(e)} = L_i \nabla L_j + L_j \nabla L_i$$

$$\vec{G}_2^{(e)} = \left[L_j \left(2L_i - L_j \right) \nabla L_i + L_i \left(L_i - 2L_j \right) \nabla L_j \right]$$

  - Face Space (2 per face): Ex: Face (i,j,k):

$$\vec{R}_{2,0}^{(f)} = L_i \vec{W}_{j,k}, \qquad \Re\left(\vec{R}_{2,0}^{(f)} \right) = L_j \vec{W}_{k,i}$$

$$\vec{G}_{2,0}^{(f)} = L_1 L_2 \nabla L_0 + L_0 L_2 \nabla L_1 + L_0 L_1 \nabla L_2$$

$$\vec{R}_{3,0}^{(f)} = L_1 L_2 \vec{W}_{2,0}, \quad \Re\left(\vec{R}_{3,0}^{(f)} \right) = L_2 L_0 \vec{W}_{0,1}, \quad \Re\Re\left(\vec{R}_{3,0}^{(f)} \right) = L_0 L_1 \vec{W}_{1,2}$$

  - Volume (3 functions)

$$\vec{R}_{3,0,0}^{(v)} = L_1 L_2 L_3 \nabla L_0$$

$$\vec{R}_{3,0,1}^{(v)} = L_2 L_3 L_0 \nabla L_1$$

$$\vec{R}_{3,0,2}^{(v)} = L_3 L_0 L_1 \nabla L_2$$

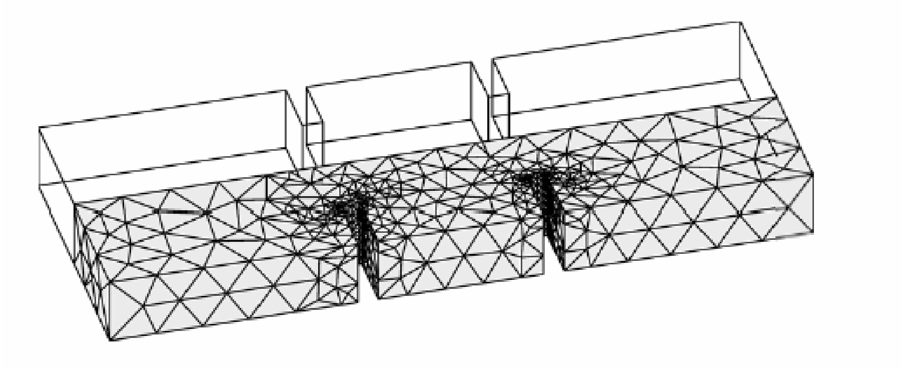
Discussion

- For the H(2)-curl function:
  - The rotational and gradient spaces (through G2 and R2) form a polynomial complete expansion up to second order for each of the three independent reciprocal vectors.
  - The rotational spaces R3 for the face and volume functions complete the rotational space to 3rd order. However, the entire function space is not complete to 3rd order.
  - The curl of rotational spaces (including H0, R2, R3) is complete to second order.

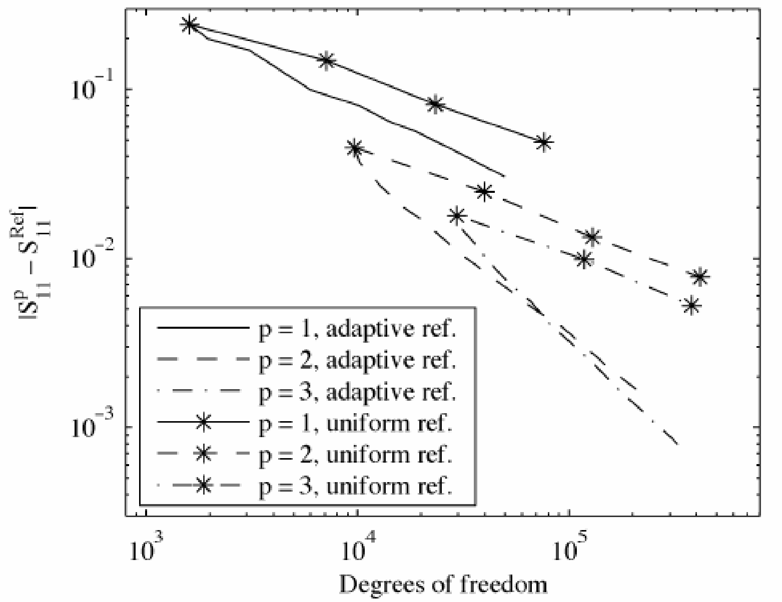
- Higher-order function spaces can be constructed in a similar manner.
- In general when using higher-order basis functions in a Finite-Element Method formulation, analytics can be used for linear tetrahedron for evaluating the local element matrices.
  - However, numerical integration using Gauss-Quadrature rules is exact, and can be very efficiently implemented
  - Can treat curvilinear tetrahedron directly as well.

Advantages of a Higher-Order Basis Function Space

- Example – Waveguide Filter:



- Error in the scattering scattering parameter S_{11}[3]



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[3] P. Ingelström, “A new set of H(curl)-conforming hierarchical basis functions for tetrahedral meshes”, *IEEE Trans. A. P.*, vol. 54, pp. 106-115, Jan. 2006