ABSORBING BOUNDARY CONDITIONS

- FEM Analysis of EM fields in Unbounded Media

- Restricted to a finite domain size. Thus, must apply a specialized boundary condition that "Absorbs" any energy impinging upon it.

- Objective:
  - Reflectionless Boundary that is independent of angle of incidence and polarization.
  - Minimal computational cost (in terms of CPU and memory)

- Possible techniques:

  - Exact radiation boundary condition based on an integral equation formulation (using Green's functions)
  - Local radiation boundary condition, or "Absorbing Boundary Condition" (ABC) based on a differential operator, or psuedo-differential operator
  - Absorbing media that is matched to the problem domain (e.g., an anechoic absorber)

Sommerfeld Radiation Condition

- Consider a plane wave propagating through a homogeneous media (e.g., TMz)

$$E_z = E_0 e^{-j(k_x x + k_y y)}$$

- Consider a planar boundary with $\hat{n} = \hat{x}$. Then

$$\frac{\partial E_z}{\partial n} = \frac{\partial E_z}{\partial x} = -jk_x E_z = -jk\cos\phi E_z$$

- Approximate:

$$\frac{\partial E_z}{\partial n} \approx -jkE_z$$

- This is the Sommerfeld Radiation Condition.

- The transverse wave impedance of the incident wave:

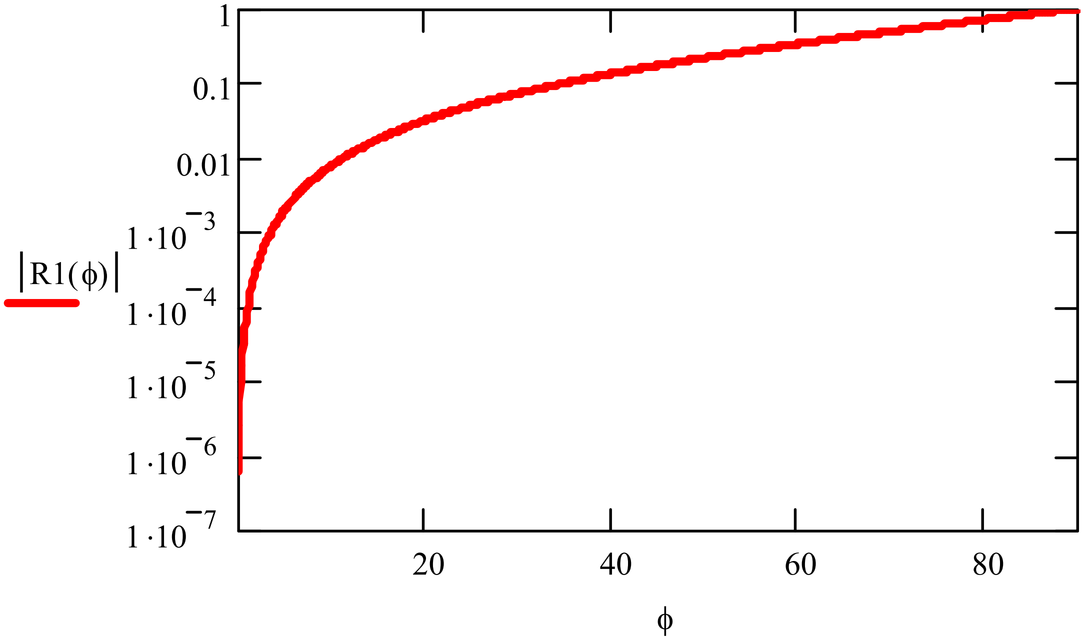
$$Z_x = \frac{E_z^{inc}}{-H_y^{abc}} = -\frac{E_z^{inc}}{\frac{1}{j\omega\mu}\frac{\partial E_z^{inc}}{\partial x}} \approx \frac{E_z^{inc}}{\frac{-1}{j\omega\mu}\left(-jk_x E_z^{inc}\right)} = \frac{\eta}{\cos\phi}$$

- The effective wave impedance of the ABC surface:

$$Z_{abc} = \frac{E_z^{inc}}{-H_y^{abc}} = -\frac{E_z^{inc}}{\frac{-1}{-j\omega\mu}\frac{\partial E_z^{inc}}{\partial n}} \approx \frac{E_z^{inc}}{\frac{-1}{j\omega\mu}\left(-jkE_z^{inc}\right)} = \eta$$

- The Reflection coefficient of the ABC surface computed via transmission-line analogy:

$$R = \frac{\eta - \frac{\eta}{\cos\phi}}{\eta + \frac{\eta}{\cos\phi}} = \frac{\cos\phi - 1}{\cos\phi + 1}$$



ENGQUIST-MAJDA ABSORBING BOUNDARY

- Again, we can write the normal derivative for $E_z = E_0 e^{-j(k_x x \pm k_y y)}$ as:

$$\frac{\partial}{\partial x} E_z = -jk_x E_z = -j\sqrt{k^2 - k_y^2}\, E_z = -jk\sqrt{1 - \frac{k_y^2}{k^2}}$$

- The objective is to translate this term into a differential operator, recognizing that $\frac{\partial}{\partial y} E_z = \mp jk_y E_z$

- Applying a Taylor series expansion:

$$\frac{\partial}{\partial x} E_z \approx -jk\left(1 - \frac{1}{2}\left(\frac{k_y}{k}\right)^2\right) E_z$$

- This can be re-written in a differential form as:

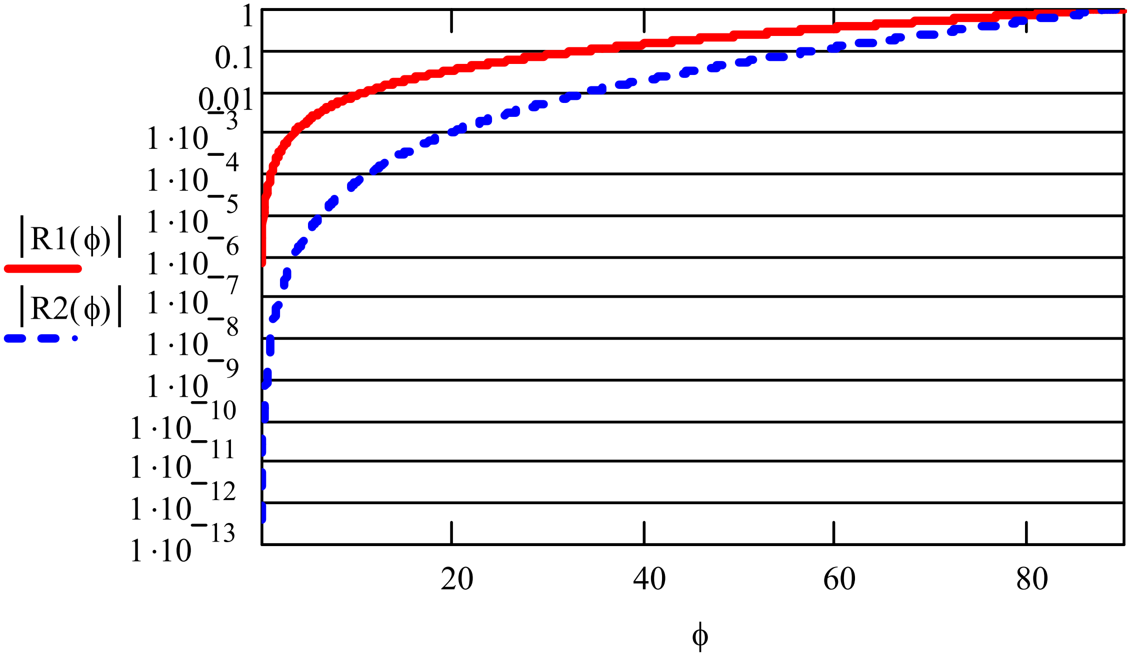
$$\frac{\partial}{\partial x} E_z \approx -jkE_z - \frac{j}{2k}\frac{\partial^2}{\partial y^2} E_z$$

[B. Engquist and A. Majda, "Absorbing boundary conditions for the numerical simulation of waves", *Mathematics of Computation*, vol. 31, 1977, pp. 629-651]

- Reflection Error:

$$Z_{abc} \approx \frac{E_z^{inc}}{\frac{-1}{j\omega\mu}\left(-jkE_z^{inc} - \frac{j}{2k}(\mp jk\sin\phi)^2 E_z^{inc}\right)} = \frac{\eta}{1-\sin^2\phi/2}$$

$$R = \frac{\frac{\eta}{1-\sin^2\phi/2} - \frac{\eta}{\cos\phi}}{\frac{\eta}{1-\sin^2\phi/2} + \frac{\eta}{\cos\phi}} = \frac{\cos\phi + \sin^2\phi/2 - 1}{\cos\phi - \sin^2\phi/2 + 1}$$



OTHER ABC OPERATORS

- The Engquist-Majda ABC annihilates a normally incident wave highly accurately. However, for obliquely incident waves, large reflections can occur.
- Other boundary operators have been introduced to annihilate waves at multiple angles. This produces a higher level of absorption over multiple angles.
- At least 2 schemes have been proposed with this intention
  - Trefethen-Halpern Generalized ABC
  - Higdon Boundary Operator.

<u>Trefethen-Halpern Approximation</u>

- Applying a Padé-type approximation:

$$\frac{\partial}{\partial x}E_z \approx -jk\frac{p_0 + p_2 s^2}{q_0 + q_2 s^2}E_z$$

- where $s = k_y / k$. This leads to:

$$\left(k^2 q_0 + q_2 k_y^2\right)\frac{\partial}{\partial x}E_z = -jk\left(p_0 k^2 + p_2 k_y^2\right)E_z$$

replacing $\pm jk_y \rightarrow \partial/\partial y$, this leads to a third-order term:

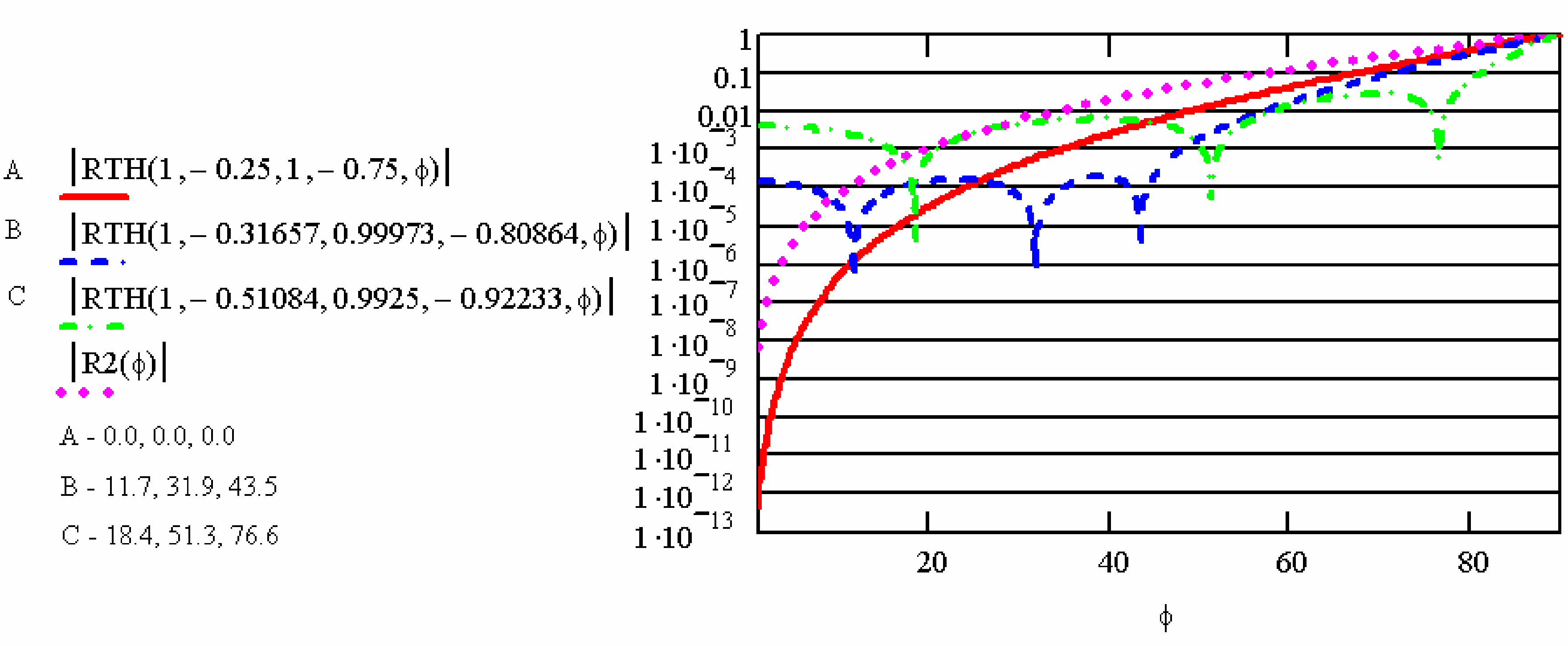
$$\left(k^2 q_0 - q_2\frac{\partial^2}{\partial y^2}\right)\frac{\partial}{\partial x}E_z = -jk\left(p_0 k^2 - p_2\frac{\partial^2}{\partial y^2}\right)E_z$$

TREFETHEN-HALPERN OPERATOR

Reflection Error:

$$R = \frac{q_0 \cos\phi + q_2 \sin\phi \sin^2\phi - p_0 - p_2 \sin^2\phi}{q_0 \cos\phi + q_2 \cos\phi \sin^2\phi + p_0 + p_2 \sin^2\phi}$$

$$\mathrm{RTH}(q0, q2, p0, p2, \phi) := \frac{q0\cdot\cos(\phi) + q2\cdot\cos(\phi)\cdot\sin(\phi)^2 - p0 - p2\cdot\sin(\phi)^2}{q0\cdot\cos(\phi) + q2\cdot\cos(\phi)\cdot\sin(\phi)^2 + p0 + p2\cdot\sin(\phi)^2}$$



HIGDON BOUNDARY OPERATOR

- Consider the Wave Impinging on the Exterior Boundary to be a Linear Superposition of Plane Waves

$$E_z(x,y) = \sum_i f_i(\cos\phi_i, \pm\sin\phi_i)$$

- Higdon Proposed the Annihilator function [*Math. Comp.*, vol. 49, pp. 65-90, 1987]:

$$\left[\prod_{i=1}^{n}\left(jk\cos\phi_i + \frac{\partial}{\partial x}\right)\right]E_z(x,y) = 0$$

- This operator is *exact* at each of the *n* angles ϕ_i.
- First-Order Higdon (*n* = 1):

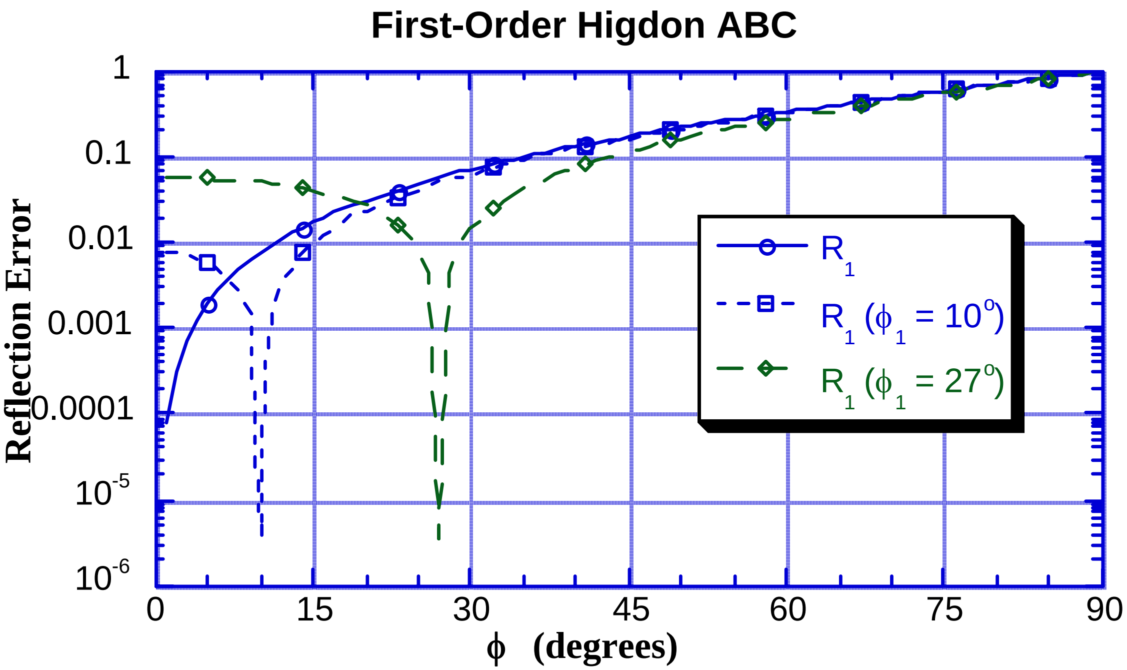
$$\left(jk\cos\phi_1 + \frac{\partial}{\partial x}\right)E_z(x,y) = 0$$

- Actually Equivalent to the First-Order Engquist-Majda, with $\frac{\partial}{\partial x}E_z \approx -jk_xE_z$

$$Z_{abc} \approx \frac{-j\omega\mu E_z^{inc}}{\left(-jk_x E_z^{inc}\right)} = \frac{\eta}{\cos\phi_1}$$

- The Reflection coefficient of the ABC surface computed via transmission-line analogy:

$$R = \frac{\dfrac{\eta}{\cos\phi_1} - \dfrac{\eta}{\cos\phi}}{\dfrac{\eta}{\cos\phi_1} + \dfrac{\eta}{\cos\phi}} = \frac{\cos\phi - \cos\phi_1}{\cos\phi + \cos\phi_1}$$



- Second-Order Higdon ($n = 1$):

$$\left(jk\cos\phi_1 + \frac{\partial}{\partial x}\right)\left(jk\cos\phi_2 + \frac{\partial}{\partial x}\right)E_z\left(x,y\right) = 0$$

- Annihilates incident waves at angles ϕ_1 and ϕ_2.

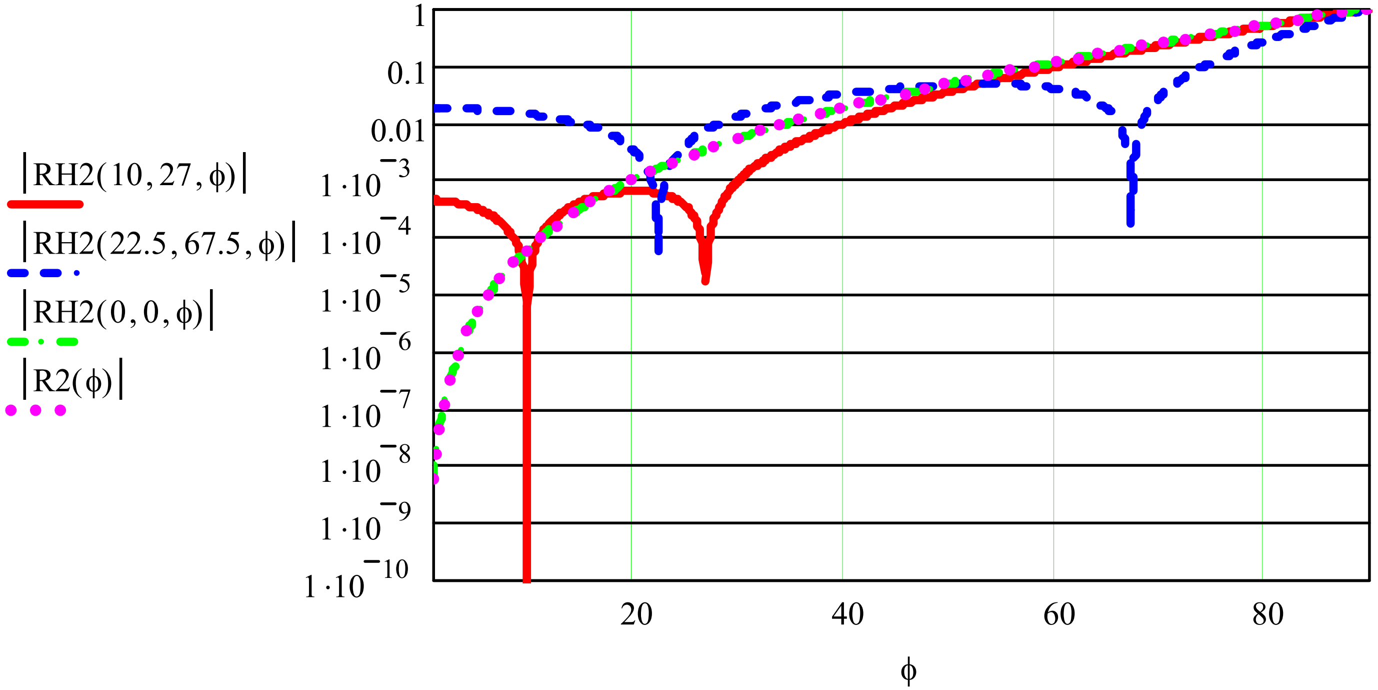
- From the wave equation: $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)E_z = 0$. Thus, this can be rewritten as:

$$jk\frac{\partial E_z}{\partial x} - k^2\frac{1+\cos\phi_1\cos\phi_2}{(\cos\phi_1+\cos\phi_2)}E_z - \frac{1}{(\cos\phi_1+\cos\phi_2)}\frac{\partial^2 E_z}{\partial y^2} = 0$$

- This leads to reflection error:

$$Z_{abc} \approx \frac{\eta}{\dfrac{1+\cos\phi_1\cos\phi_2}{(\cos\phi_1+\cos\phi_2)} - \dfrac{\sin^2\phi}{(\cos\phi_1+\cos\phi_2)}}$$

$$R = \frac{\cos\phi(\cos\phi_1+\cos\phi_2) - \cos^2\phi - \cos\phi_1\cos\phi_2}{\cos\phi(\cos\phi_1+\cos\phi_2) + \cos^2\phi + \cos\phi_1\cos\phi_2}$$


|RH2(10, 27, ϕ)|
|RH2(22.5, 67.5, ϕ)|
|RH2(0, 0, ϕ)|
|R2(ϕ)|
1
0.1
0.01
1·10^−3
1·10^−4
1·10^−5
1·10^−6
1·10^−7
1·10^−8
1·10^−9
1·10^−10
20
40
60
80
ϕ

IMPLEMENTING THE ABC'S

- The ABC's only affect the boundary integral:

$$\int_{\partial\Omega} E_z^a \frac{1}{\mu_r} \frac{\partial E_z}{\partial n} d\ell$$

Sommerfeld Radiation Condition:

$$\left. \frac{\partial E_z}{\partial n} \right|_{\partial\Omega_{ABC}} \approx -jkE_z$$

- Assuming Nodal Basis functions:

$$E_z \approx \sum_{j=1}^{n} c_j N_j; \quad E_z^a = N_i$$

$$\int_{\partial\Omega} E_z^a \frac{1}{\mu_r} \frac{\partial E_z}{\partial n} d\ell \approx -jk \sum_{j=1}^{n} c_j \int_{\partial\Omega} N_i \frac{1}{\mu_r} N_j d\ell$$

SECOND-ORDER ENGQUIST-MAJDA ABC

$$\frac{\partial}{\partial x}E_z \approx -jkE_z - \frac{j}{2k}\frac{\partial^2}{\partial y^2}E_z$$

On an x-normal boundary:

$$\int_{\partial\Omega} E_z^a \frac{1}{\mu_r}\frac{\partial E_z}{\partial x}dy \approx -jk\int_{\partial\Omega} E_z^a \frac{1}{\mu_r}E_z dy - \frac{j}{2k}\int_{\partial\Omega} E_z^a \frac{1}{\mu_r}\frac{\partial^2}{\partial y^2}E_z dy$$

Applying integration by parts:

$$\int_{\partial\Omega} E_z^a \frac{1}{\mu_r}\frac{\partial E_z}{\partial x}dy \approx -jk\int_{\partial\Omega} E_z^a \frac{1}{\mu_r}E_z dy + \frac{j}{2k}\int_{\partial\Omega} \frac{\partial}{\partial y}E_z^a \frac{1}{\mu_r}\frac{\partial}{\partial y}E_z dy$$

This is generalized to a boundary of arbitrary normal:

$$\int_{\partial\Omega} E_z^a \frac{1}{\mu_r}\frac{\partial E_z}{\partial n}d\ell \approx -jk\int_{\partial\Omega} E_z^a \frac{1}{\mu_r}E_z d\ell + \frac{j}{2k}\int_{\partial\Omega} \frac{\partial}{\partial t}E_z^a \frac{1}{\mu_r}\frac{\partial}{\partial t}E_z d\ell$$

where $\partial/\partial t$ is the transverse derivative.

SECOND-ORDER HIGDON ABC

$$\frac{\partial E_z}{\partial x} \approx -jk\frac{1+\cos\phi_1\cos\phi_2}{(\cos\phi_1+\cos\phi_2)}E_z - \frac{j}{k}\frac{1}{(\cos\phi_1+\cos\phi_2)}\frac{\partial^2 E_z}{\partial y^2}$$

Following the 2$^{\text{nd}}$-order Engquist-Majda implementation, this is implemented as:

$$\int_{\partial\Omega} E_z^a \frac{1}{\mu_r}\frac{\partial E_z}{\partial n}d\ell \approx -jk\frac{1+\cos\phi_1\cos\phi_2}{(\cos\phi_1+\cos\phi_2)}\int_{\partial\Omega} E_z^a\frac{1}{\mu_r}E_z d\ell$$
$$+\frac{j}{k}\frac{1}{(\cos\phi_1+\cos\phi_2)}\int_{\partial\Omega}\frac{\partial}{\partial t}E_z^a\frac{1}{\mu_r}\frac{\partial}{\partial t}E_z d\ell$$

Again, assuming Nodal elements this is implemented as:

$$\int_{\partial\Omega} E_z^a \frac{1}{\mu_r}\frac{\partial E_z}{\partial n}d\ell \approx -\frac{jk}{\mu_r}K_1\sum_{j=1}^{n}c_j\int_{\partial\Omega}N_iN_j d\ell + \frac{j}{k\mu_r}K_2\sum_{j=1}^{n}c_j\int_{\partial\Omega}\frac{\partial N_i}{\partial t}\frac{\partial N_j}{\partial t}d\ell$$

where, K_1 and K_2 are constants as seen above. Note that in the second term, nodes off the boundary will contribute to the term, even though the integral is over the boundary edge.

THIRD-ORDER TREFETHEN-HALPERN ABC

$$\left(k^2 q_0 - q_2 \frac{\partial^2}{\partial t^2} \right) \frac{\partial}{\partial n} E_z = -jk \left(p_0 k^2 - p_2 \frac{\partial^2}{\partial t^2} \right) E_z$$

The 3rd-order Trefethen-Halpern ABC is not as easily implemented b/c of the mixed-derivatives. However, it can be implemented even with first-order elements following a method proposed by O. Ramahi [IEEE Trans. On AP, Vol AP-47, pp. 1141-1145, 1999].

Let: $\Phi^b = \partial E_z / \partial n$.

Expand: $\Phi^b = \sum_{i=1}^{N_b} b_i N_i^b$, where N_b are the number of nodes on the boundary, and N_i^b are the basis functions on the boundary only.

Also, let $E_z^b = \sum_{i=1}^{N_b} c_i N_i^b$, be an expansion of E_z restricted to the boundary.

Then, we apply the condition: $\left(k^2 q_0 - q_2 \frac{\partial^2}{\partial t^2} \right) \Phi^b = -jk \left(p_0 k^2 - p_2 \frac{\partial^2}{\partial t^2} \right) E_z^b$

This leads to the relationship:

$$\sum_{j=1}^{n} b_j \left[k^2 q_0 \int_{\partial\Omega} N_i^b N_j^b d\ell + q_2 \int_{\partial\Omega} \frac{\partial N_i^b}{\partial t} \frac{\partial N_j^b}{\partial t} d\ell \right] =$$

$$-jk \sum_{k=1}^{n} c_k \left[k^2 p_0 \int_{\partial\Omega} N_l^b N_k^b d\ell + p_2 \int_{\partial\Omega} \frac{\partial N_l^b}{\partial t} \frac{\partial N_k^b}{\partial t} d\ell \right]$$

This can be represented via a linear system of equations, as:

$$[Q]\overline{b} = [P]\overline{c}^b$$

where, $\overline{c}^b$ are restricted to nodes on the ABC boundary.

We then formulate the ABC boundary integral as:

$$\int_{\partial\Omega} E_z^a \frac{1}{\mu_r} \frac{\partial E_z}{\partial n} d\ell = \int_{\partial\Omega} E_z^a \frac{1}{\mu_r} \Phi d\ell \approx \sum_{j=1}^{n} b_j \int_{\partial\Omega} N_i \frac{1}{\mu_r} N_j^b d\ell$$

which contributes to the global *K*-matrix. The two systems are solved simultaneously, leading to the solution for the c_j.

2D ABC FOR A CIRCULAR OUTER BOUNDARY

- Asymptotic approximation of a two-dimensional far-field:

$$E_z \approx A(\phi)\frac{e^{-jk\rho}}{\sqrt{\rho}}$$

- First-order approximation of the normal derivative:

$$\frac{\partial E_z}{\partial \rho} \approx \left(-jk - \frac{1}{2\rho}\right) A(\phi)\frac{e^{-jk\rho}}{\sqrt{\rho}} = \left(-jk - \frac{1}{2\rho}\right) E_z$$

- First-order (Sommerfeld) Radiation Condition on a circular cylindrical boundary:

$$\boxed{\frac{\partial E_z}{\partial \rho} \approx \left(-jk - \frac{1}{2\rho}\right) E_z}$$

HIGHER-ORDER EXPANSION

- Wilcox far-field expansion:

$$E_z \approx \frac{e^{-jk\rho}}{\sqrt{\rho}} \sum_{n=0}^{\infty} \frac{a_n(\phi)}{\rho^n}$$

- Bayliss-Gunzberger-Turkel annihilation operator of order *m*:

$$\mathfrak{B}_m = \left(\frac{\partial}{\partial\rho} + jk + \frac{4m-3}{2\rho} \right) \mathfrak{B}_{m-1}$$

where $\mathfrak{B}_0 = 1$.

Can show that: $\mathfrak{B}_m(E_z) = O\left(\rho^{-2m-1/2}\right)$. Thus, $\mathfrak{B}_m$ annihilates the first *m* terms of the Wilcox expansion series.

- $\mathfrak{B}_1$-operator:

$\mathfrak{B}_1 E_z = 0$

$$\left(\frac{\partial}{\partial\rho} + jk + \frac{1}{2\rho} \right) E_z \approx 0 \quad \Rightarrow \quad \frac{\partial}{\partial\rho} E_z \approx -\left(jk + \frac{1}{2\rho} \right) E_z$$

- $\mathfrak{B}_2$-operator:

$$\mathfrak{B}_2 E_z = 0$$

$$\left(\frac{\partial}{\partial\rho} + jk + \frac{5}{2\rho}\right)\left(\frac{\partial}{\partial\rho} + jk + \frac{1}{2\rho}\right) E_z = 0$$

Expanding:

$$\left(\frac{\partial^2}{\partial\rho^2} + jk2\frac{\partial}{\partial\rho} - \frac{1}{2\rho^2} - k^2 + \frac{jk}{2\rho} + \frac{5}{2\rho}\frac{\partial}{\partial\rho} + \frac{jk5}{2\rho} + \frac{5}{4\rho^2}\right) E_z \approx 0$$

Realizing that: $\frac{\partial^2}{\partial\rho^2} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - \frac{1}{\rho}\frac{\partial}{\partial\rho}$, and from the wave-equation in cylindrical coordinates: $\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + k^2\right) E_z = 0$, then:

$$\frac{\partial^2}{\partial\rho^2} = -\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} - k^2 - \frac{1}{\rho}\frac{\partial}{\partial\rho}$$

Then, expanding the above operator, and applying this relationship, leads to:

$$\frac{\partial E_z}{\partial\rho} + \left[jk + \frac{1}{2\rho} - \frac{1}{8\rho^2(1/\rho + jk)}\right] E_z - \frac{1}{2\rho^2(1/\rho + jk)}\frac{\partial^2 E_z}{\partial\phi^2} \approx 0$$

Thus, for the $\mathfrak{B}_2$-operator:

$$\boxed{\frac{\partial E_z}{\partial \rho} \approx -\left[jk + \frac{1}{2\rho} - \frac{1}{8\rho^2(1/\rho + jk)} \right] E_z + \frac{1}{2\rho^2(1/\rho + jk)} \frac{\partial^2 E_z}{\partial \phi^2}}$$

This is generalized to an arbitrary boundary with curvature $\kappa = 1/\rho$. Realizing:

$$\frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} \to \frac{\partial^2}{\partial t^2}$$

Then, the $\mathfrak{B}_2$-operator is expressed as:

$$\boxed{\frac{\partial E_z}{\partial n} \approx -\left[jk + \frac{\kappa}{2} - \frac{\kappa^2}{8(\kappa + jk)} \right] E_z + \frac{1}{2(\kappa + jk)} \frac{\partial^2 E_z}{\partial t^2}}$$

Assuming Nodal elements this is implemented as:

$$\int_{\partial\Omega} E_z^a \frac{1}{\mu_r}\frac{\partial E_z}{\partial n} d\ell \approx -\frac{1}{\mu_r}\left[jk + \frac{\kappa}{2} - \frac{\kappa^2}{8(\kappa + jk)} \right] \sum_{j=1}^{n} c_j \int_{\partial\Omega} N_i N_j d\ell$$

$$-\frac{1}{2(\kappa + jk)} \frac{1}{\mu_r} \sum_{j=1}^{n} c_j \int_{\partial\Omega} \frac{\partial N_i}{\partial t}\frac{\partial N_j}{\partial t} d\ell$$

Note, if $\kappa = 0$, this reduces exactly to the 2nd-Order Engquist-Majda ABC.

ABC's FOR VECTOR FORMULATIONS IN 3D

- Vector Formulation Boundary Integral:

$$\int_{\partial\Omega} \vec{E}^a \times \frac{1}{\mu_r} \nabla \times \vec{E} \cdot \hat{n} ds$$

where $\hat{n}$ is the outward normal to $\partial\Omega$, and $\vec{E}$ satisfies the vector-Helmholtz equation:

$$\nabla \times \nabla \times \vec{E} - k^2 \vec{E} = 0$$

- Wilcox Expansion of the far-field in 3D spherical coordinates:

$$\vec{E} = \frac{e^{-jkr}}{r} \sum_{n=0}^{\infty} \frac{\vec{A}_n(\theta,\phi)}{r^n}$$

- Define the Webb-Kanellopoulos operator [MOTL, vol. 2, pp. 370-372, Oct. 1989]:

$$\mathfrak{L}_{\mathfrak{m}}(\vec{u}) = \hat{r} \times \nabla \times \vec{u} - \left(jk + \frac{m}{r} \right) \vec{u}$$

- It can be shown that for $m \geq 0$ and $n \geq 0$:

$$\mathfrak{L}_{\mathfrak{m}}\left(e^{-jkr} \frac{\hat{r} \times \vec{A}_n(\theta,\phi)}{r^{n+1}} \right) = (n-m) e^{-jkr} \frac{\hat{r} \times \vec{A}_n(\theta,\phi)}{r^{n+2}}$$

- And for $m \geq 0$ and $n \geq 0$:

$$\mathfrak{L}_{\mathfrak{m}}\left[\nabla_t\left(e^{-jkr}\frac{\hat{r}\cdot\vec{A}_n(\theta,\phi)}{r^{n+1}}\right)\right]=(n+1-m)\nabla_t\left(e^{-jkr}\frac{\hat{r}\cdot\vec{A}_n(\theta,\phi)}{r^{n+2}}\right)$$

where, ∇_t is the grad operator transverse to the radial direction.

- What is interesting, is that in both cases, the $\mathfrak{L}_{\mathfrak{m}}$ operator has the affect of multiplying the function by 1/r times an integer (n-m) or ($n + 1 - m$).

- Webb-Kanellopoulos ABC operator:

$$\mathfrak{B}_{\mathfrak{m}}(\vec{u})=(\mathfrak{L}_{\mathfrak{m}})^{m}(\vec{u}_t)+s(\mathfrak{L}_{\mathfrak{m}})^{m-1}(\nabla_t u_r)$$

where $(\mathfrak{L}_{\mathfrak{m}})^{m}$ implies applying the operator successively m-times, and s is a constant to be determined, $\vec{u}_t$ is the transverse part of $\vec{u}$, and u_r is the radial part of $\vec{u}$.

- One can show that:

$$\mathfrak{B}_{\mathfrak{m}}(\vec{E})=O\left(r^{-(2m+1)}\right)$$

THE WEBB-KANELLOPOULOS VECTOR-ABC

- **First-order WK-ABC:**

$$\mathfrak{B}_1\left(\vec{E}\right) = \hat{r}\times\nabla\times\vec{E}_t - jk\vec{E}_t + s\nabla_t\hat{r}\cdot\vec{E} \approx 0$$

- Note that we have to cast $\hat{r}\times\nabla\times\vec{E}_t = \hat{r}\times\nabla\times\vec{E} - \hat{r}\times\nabla\times\hat{r}E_r$.

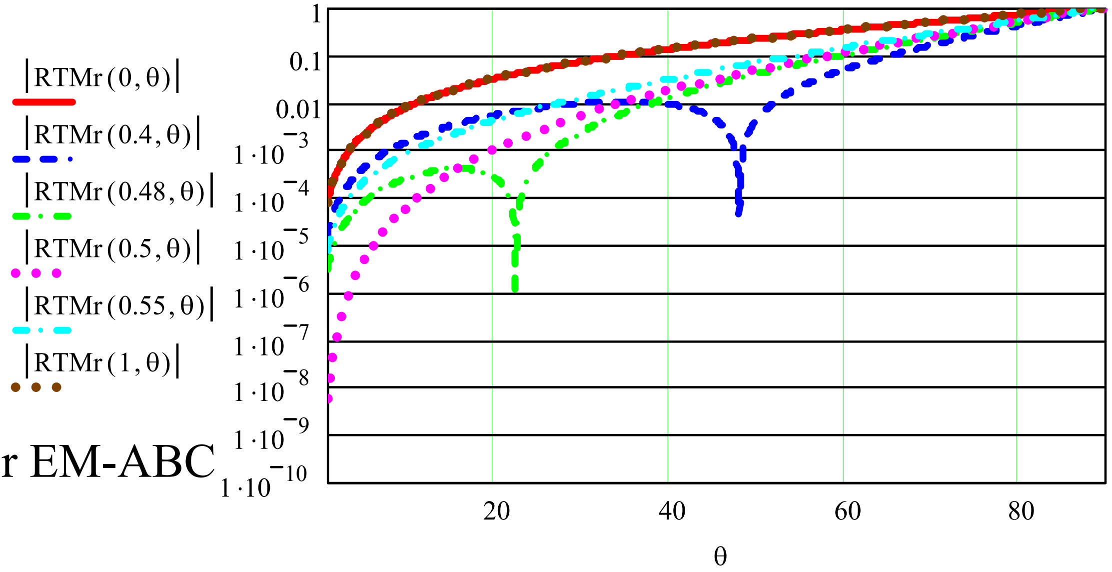
- Can show that: $\hat{r}\times\nabla\times\hat{r}E_r = -\hat{r}\times\hat{r}\times\nabla E_r = \nabla_t E_r$. Therefore, from the $\mathfrak{B}_1$-operator:

$$\hat{r}\times\nabla\times\vec{E} \approx jk\vec{E}_t - \left(s-1\right)\nabla_t E_r$$

- Reflection Error:

$$R_1^{TE_r} = \frac{\cos\theta - 1}{\cos\theta + 1}$$

$$R_1^{TM_r} = \frac{\cos\theta + \left(1-s\right)\sin^2\theta - 1}{\cos\theta - \left(1-s\right)\sin^2\theta + 1}$$



Note: if $s = 1/2 \Rightarrow$ same R as 2^{nd}-order EM-ABC

- **Second-order WK-ABC:**

$$\mathfrak{B}_2\left(\vec{E}\right) = \mathfrak{L}_1\mathfrak{L}_1\left(\vec{E}_t\right) + s\mathfrak{L}_2\left(\nabla_t E_r\right)$$

$$= -2\left(jk + \frac{1}{r}\right)\hat{r}\times\nabla\times\vec{E} + 2jk\left(jk + \frac{1}{r}\right)\vec{E}_t + \nabla\times\hat{r}\left(\hat{r}\cdot\nabla\times\vec{E}\right)$$

$$+\left(s-1\right)\nabla_t\left(\nabla\cdot\vec{E}_t\right) + \left(2-s\right)jk\nabla_t E_r$$

Therefore, from the $\mathfrak{B}_2$-operator:

$$\hat{r}\times\nabla\times\vec{E} \approx jk\vec{E}_t + \frac{r}{2\left(jkr+1\right)}\left[\nabla\times\hat{r}\left(\hat{r}\cdot\nabla\times\vec{E}\right) + \left(s-1\right)\nabla_t\left(\nabla\cdot\vec{E}_t\right) + \left(2-s\right)jk\nabla_t E_r\right]$$

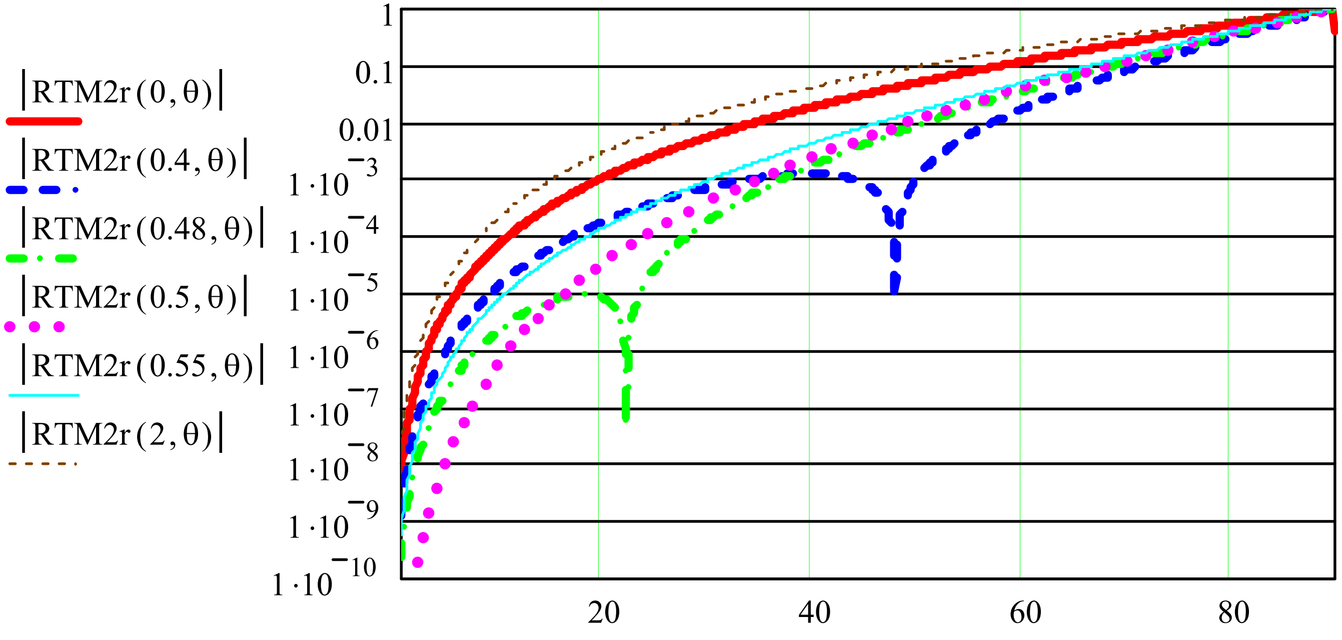
Reflection Error of the $\mathfrak{B}_2$-operator:

$$R_2^{TE_r} = \frac{\cos\theta + \sin^2\theta/2 - 1}{\cos\theta - \sin^2\theta/2 + 1}$$

$$R_2^{TM_r} = \frac{\cos\theta + (s-1)\sin^2\theta\cos\theta/2 + (1-s/2)\sin^2\theta - 1}{\cos\theta + (s-1)\sin^2\theta\cos\theta/2 - (1-s/2)\sin^2\theta + 1}$$

Observations:

$s = 1$: $R_2^{TM_r} = R_2^{TE_r}$ $s = 1/2$: $R_2^{TM_r} =$ 3rd-order ABC



IMPLEMENETING THE WK VECTOR-ABC

- **First-order WK-ABC (*s* = 1):**

$$\hat{r}\times\nabla\times\vec{E}\approx jk\vec{E}_t$$

where, $\vec{E}_t=-\hat{r}\times\hat{r}\times\vec{E}$. This is the Sommerfeld Radiation Condition.

Boundary integral term:

$$\int\limits_{\partial\Omega}\vec{E}^a\times\frac{1}{\mu_r}\nabla\times\vec{E}\cdot\hat{n}ds=\frac{1}{\mu_r}\int\limits_{\partial\Omega}\vec{E}^a\times\nabla\times\vec{E}\cdot\hat{r}ds=\frac{-1}{\mu_r}\int\limits_{\partial\Omega}\hat{r}\times\nabla\times\vec{E}\cdot\vec{E}^a ds$$

Applying the Sommerfeld Radiation Condition:

$$\frac{-1}{\mu_r}\int\limits_{\partial\Omega}\hat{r}\times\nabla\times\vec{E}\cdot\vec{E}^a ds\approx\frac{1}{\mu_r}\int\limits_{\partial\Omega}\hat{r}\times\left(\hat{r}\times\hat{r}\times\vec{E}\right)\cdot\vec{E}^a ds=\frac{-1}{\mu_r}\int\limits_{\partial\Omega}\left(\hat{r}\times\vec{E}^a\right)\cdot\left(\hat{r}\times\hat{r}\times\vec{E}\right)ds$$

$$=\frac{1}{\mu_r}\int\limits_{\partial\Omega}\left(\hat{r}\times\vec{E}^a\right)\cdot\left(\hat{r}\times\vec{E}\right)ds$$

This can be generalized to an arbitrary boundary:

$$\int_{\partial\Omega} \vec{E}^a \times \frac{1}{\mu_r} \nabla \times \vec{E} \cdot \hat{n} ds \approx \frac{1}{\mu_r} \int_{\partial\Omega} \left(\hat{n} \times \vec{E}^a\right) \cdot \left(\hat{n} \times \vec{E}\right) ds$$

Assuming vector edge elements, this leads to a contribution in the system matrix:

$$\frac{1}{\mu_r} \sum_{j=1}^{N_b} c_j \int_{\partial\Omega} \left(\hat{n} \times \vec{W}_i\right) \cdot \left(\hat{n} \times \vec{W}_j\right) ds$$

IMPLEMENETING THE WK VECTOR-ABC (cont'd)

- **Second-order WK-ABC** ($s = 2$)**:**

$$\hat{r}\times\nabla\times\vec{E} \approx -jk\hat{r}\times\hat{r}\times\vec{E} + \frac{r}{2\left(jkr+1\right)}\left[\nabla\times\hat{r}\left(\hat{r}\cdot\nabla\times\vec{E}\right)+\nabla_t\left(\nabla\cdot\vec{E}_t\right)\right]$$

Useful Identities:

1. $\vec{A}\cdot\hat{r}\times\hat{r}\times\vec{E} = -\left(\hat{r}\times\vec{A}\right)\cdot\left(\vec{r}\times\vec{E}\right) = \vec{E}\cdot\hat{r}\times\left(\hat{r}\times\vec{A}\right)$

2.

$$\begin{aligned}\vec{A}\cdot\nabla\times\hat{r}\left(\hat{r}\cdot\nabla\times\vec{E}\right) &= \nabla\cdot\left[\hat{r}\left(\nabla\times\vec{E}\right)_r\times\vec{E}^a\right]+\hat{r}\left(\nabla\times\vec{E}\right)_r\cdot\left(\nabla\times\vec{E}^a\right)\\ &= \nabla\cdot\left[\vec{E}\times\hat{r}\left(\nabla\times\vec{E}^a\right)_r-\vec{E}^a\times\hat{r}\left(\nabla\times\vec{E}\right)_r\right]+\vec{E}\cdot\nabla\times\left[\hat{r}\left(\nabla\times\vec{E}^a\right)_r\right]\end{aligned}$$

3.

$$\begin{aligned}\vec{E}^a\cdot\nabla_t\left(\nabla\cdot\vec{E}_t\right) &= \nabla\cdot\left(\vec{E}_t^a\nabla\cdot\vec{E}_t\right)-\left(\nabla\cdot\vec{E}_t^a\right)\left(\nabla\cdot\vec{E}_t\right)\\ &= \nabla\cdot\left(\vec{E}_t^a\nabla\cdot\vec{E}_t-\vec{E}_t\nabla\cdot\vec{E}_t^a\right)+\vec{E}\cdot\nabla_t\left(\nabla\cdot\vec{E}_t^a\right)\end{aligned}$$

Boundary Integral Term:

$$\frac{1}{\mu_r}\int_{\partial\Omega}\vec{E}^a\times\nabla\times\vec{E}\cdot\hat{r}ds\approx\frac{1}{\mu_r}\int_{\partial\Omega}\left[jk\vec{E}_t^a\cdot\vec{E}_t+\beta(r)\left(\nabla\times\vec{E}^a\right)_r\left(\nabla\times\vec{E}\right)_r-\beta(r)\nabla\cdot\vec{E}_t^a\nabla\cdot\vec{E}_t\right]ds$$

where,

$$\beta(r)=\frac{r}{2(jkr+1)}$$

This can be generalized to a general boundary with curvature κ, as:

$$\frac{1}{\mu_r}\int_{\partial\Omega}\vec{E}^a\times\nabla\times\vec{E}\cdot\hat{n}ds\approx\frac{1}{\mu_r}\int_{\partial\Omega}\left[jk\vec{E}_t^a\cdot\vec{E}_t+\beta(\kappa)\left(\nabla\times\vec{E}^a\right)_n\left(\nabla\times\vec{E}\right)_n-\beta(\kappa)\nabla\cdot\vec{E}_t^a\nabla\cdot\vec{E}_t\right]ds$$

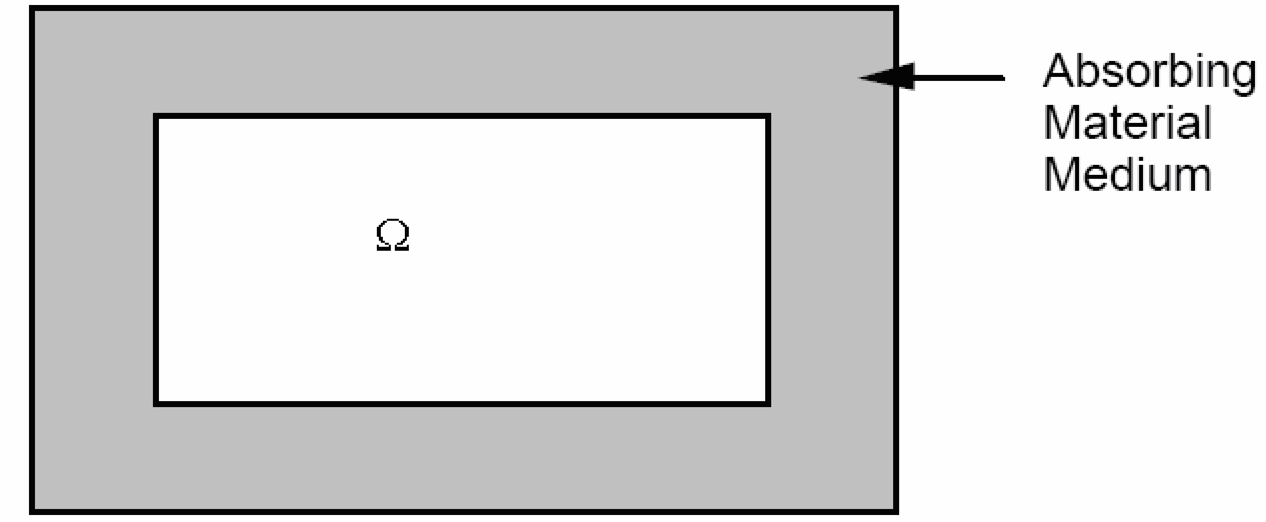
where,

$$\beta(\kappa)=\frac{1}{2(jk+\kappa)}$$

For a planar boundary, $\beta(0)=\frac{-j}{2k}$

PERFECTLY MATCHED LAYER ABC

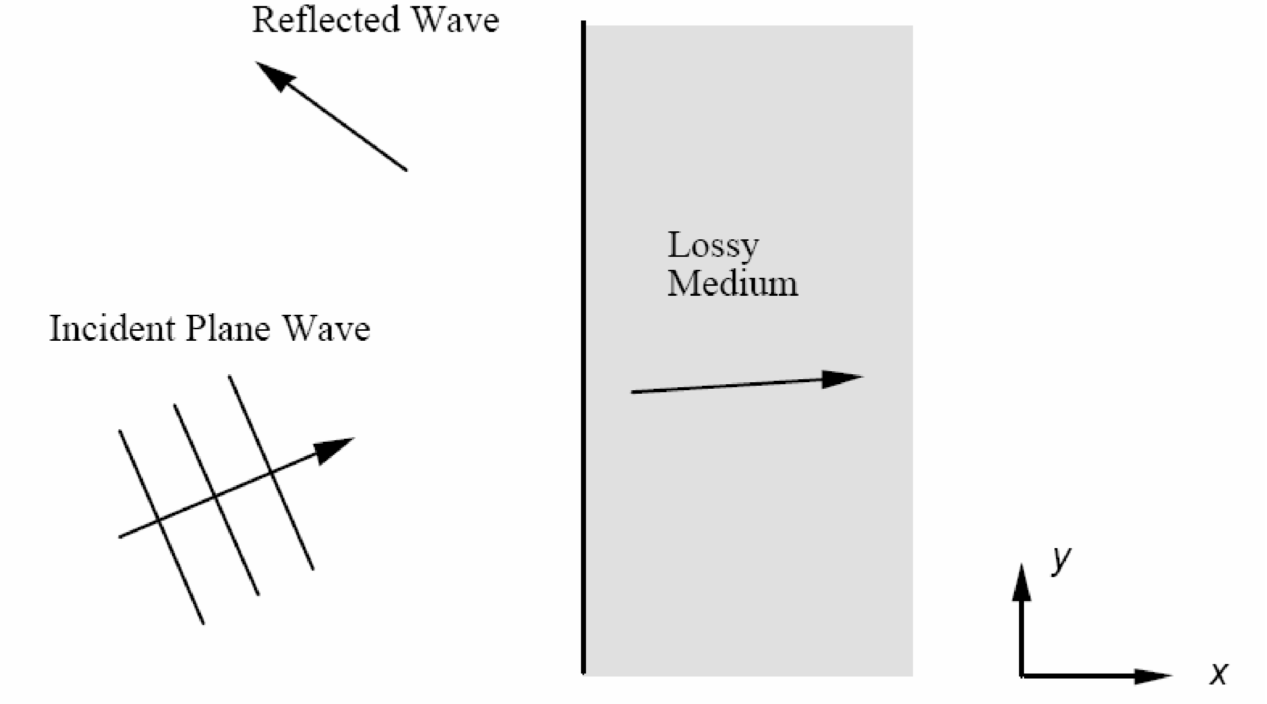
- Absorbing Material Boundary Condition'
- To Be Effective, Must:
  - Be thin (only a few lattice cells in thickness)
  - Be Effective in the Near Field of Source or Scatterer
  - Reflectionless to All Impinging Waves (Polarization, Angle)
  - Reflectionless over a Broad-Band



- Perfectly Matched Layer (PML) Material Medium First Introduced by J. P. Berenger Has Been Found to Meet all of these Criterion [J.-P. Berenger, "A perfectly matched layer for the absorption of electromagnetic waves," *Journal of Computational Physics*, October 1994]

Dispersionless Medium

- Consider a plane wave incident from a free space ($x < 0$) onto a material half space ($x > 0$)
- The material half space is assumed to have both a magnetic conductivity σ^* and an electric conductivity σ.



- TE$_z$ polarized uniform plane wave incident upon this interface $\vec{H}^{inc} = \hat{z}H_o e^{-j\beta_x^i x - j\beta_y^i y}$, propagating with angle θ relative to the normal x-axis.

- The fields in the two regions are then posed as:

$$\left.\begin{aligned}\vec{H}_1 &= \hat{z}H_o(1+\Gamma e^{2j\beta_x^i x})e^{-j\beta_x^i x - j\beta_y^i y}\\ \vec{E}_1 &= \left(-\hat{x}\tfrac{\beta_y^i}{\omega\varepsilon_1}(1+\Gamma e^{2j\beta_x^i x}) + \hat{y}\tfrac{\beta_x^i}{\omega\varepsilon_1}(1-\Gamma e^{2j\beta_x^i x})\right)H_o e^{-j\beta_x^i x - j\beta_y^i y}\end{aligned}\right\} (x<0)$$

$$\left.\begin{aligned}\vec{H}_2 &= \hat{z}H_o\tau e^{-j\beta_x^t x - j\beta_y^t y}\\ \vec{E}_2 &= \left(-\hat{x}\frac{\beta_y^t}{\omega\varepsilon_2\left(1+\frac{\sigma}{j\omega\varepsilon_2}\right)} + \hat{y}\frac{\beta_x^t}{\omega\varepsilon_2\left(1+\frac{\sigma}{j\omega\varepsilon_2}\right)}\right)H_o\tau e^{-j\beta_x^t x - j\beta_y^t y}\end{aligned}\right\} (x>0)$$

- From the dispersion relationships:

$$\left.\begin{aligned}\beta_x^i &= k_1\cos\theta^i\\ \beta_y^i &= k_1\sin\theta^i\end{aligned}\right\} (x<0)$$

$$\left.\beta_x^t = \sqrt{k_2^2\left(1+\frac{\sigma}{j\omega\varepsilon_2}\right)\left(1+\frac{\sigma *}{j\omega\mu_2}\right) - \beta_y^{t\,2}}\right\} (x>0)$$

- where $k_i = \omega\sqrt{\varepsilon_i\mu_i}$ $(i = 1{,}2)$.
- Enforcing the continuity of the tangential fields across the boundary interface ($x = 0$), $\beta_y^t = \beta_y^i = k_1\sin\theta^i$, and:

$$\Gamma = \frac{\frac{\beta_x^i}{\omega\varepsilon_1} - \frac{\beta_x^t}{\omega\varepsilon_2\left(1+\frac{\sigma}{j\omega\varepsilon_2}\right)}}{\frac{\beta_x^i}{\omega\varepsilon_1} + \frac{\beta_x^t}{\omega\varepsilon_2\left(1+\frac{\sigma}{j\omega\varepsilon_2}\right)}}; \qquad \tau = 1+\Gamma$$

- If the wave is normally incident ($\theta^i = 0$):

$$\Gamma = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}$$

where

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}, \quad \eta_2 = \sqrt{\frac{\mu_2\left(1+\frac{\sigma^*}{j\omega\mu_2}\right)}{\varepsilon_2\left(1+\frac{\sigma}{j\omega\varepsilon_2}\right)}}$$

Subsequently, if $\mu_2 = \mu_1$, $\varepsilon_2 = \varepsilon_1$, and

$$\frac{\sigma^*}{\mu_1} = \frac{\sigma}{\varepsilon_1}$$

then, $\Gamma = 0$!

- Also,

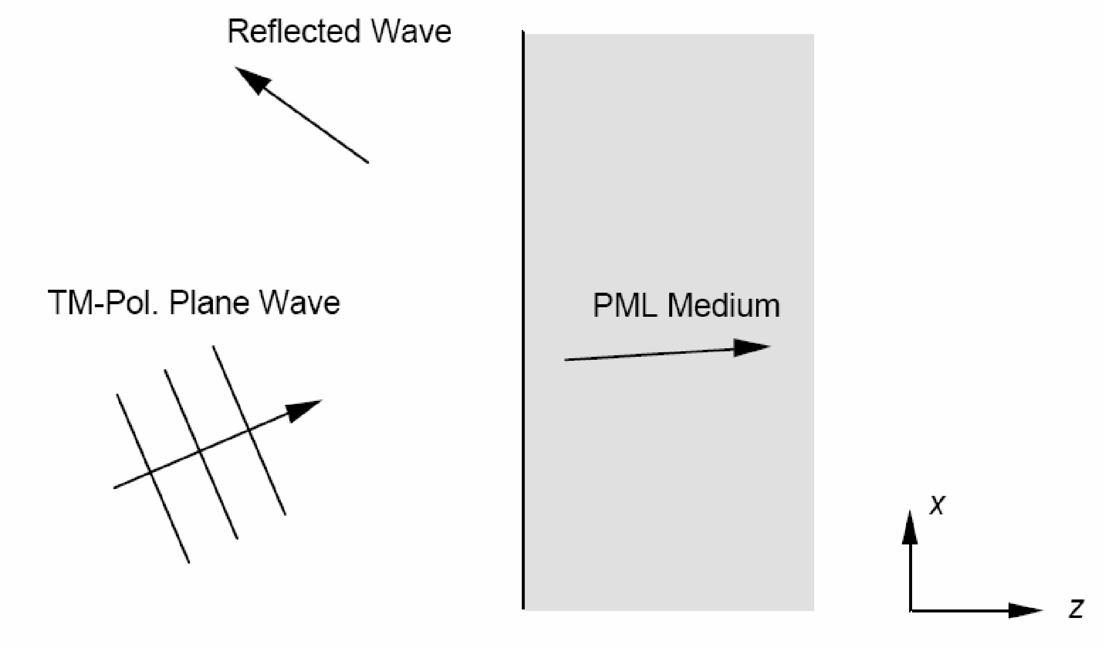
$$\beta_x^t = \left(1+\frac{\sigma}{j\omega\varepsilon_1}\right)k_1 = k_1 - j\sigma\eta_1.$$

- The real part of β_x^t is the propagating component of the wave
- The imaginary part is the attenuative component.
- Thus, the wave speed in medium 2 is identical to that in medium 1. Furthermore, despite being a lossy medium, the wave is dispersionless – i.e., the wave speed is frequency independent for an axially propagating wave. Hence, this type of medium has been quantified as a dispersionless medium.
- From (7.1.b), the resultant fields in the dispersionless medium excited by a normally incident wave are expressed as:

$$\vec{H}_2 = \hat{z}H_o e^{-jk_1x} e^{-\sigma\eta_1 x}$$
$$\vec{E}_2 = \hat{y}\eta_1 H_o e^{-jk_1x} e^{-\sigma\eta_1 x}.$$

- In summary, it is seen that given a medium with magnetic and electric conductivity defined by $\frac{\sigma *}{\mu_1} = \frac{\sigma}{\varepsilon_1}$ the medium will be matched across a planar boundary for all *normally* incident waves. Furthermore, the wave propagating in the dispersionless medium has the same propagation characteristics as the incident wave, but will attenuate along the normal direction.

UNIAXIAL PERFECTLY MATCHED LAYER ABSORBING MEDIUM



- Assume a time-harmonic arbitrarily polarized plane wave, incident on a material half space described as a uniaxial anisotropic medium:

$$\vec{H}^{inc} = \vec{H}_o e^{-j\beta_x^i x - j\beta_z^i z}$$

† Z. S. Sacks, D. M. Kingsland, R. Lee, and J. F. Lee, "A perfectly matched anisotropic absorber for use as an absorbing boundary condition," *IEEE Transactions on Antennas and Propagation* , vol. 43, pp. 1460-1463, December 1995.

†† S. D. Gedney, "An Anisotropic PML Absorbing Media for FDTD Simulation of Fields in Lossy Dispersive Media," *Electromagnetics*, vol. 16, no. 3, July/August, 1996.

- The interface between the two media is the $z = 0$ plane.
- The fields excited within the uniaxial medium are plane wave in nature and satisfy Maxwell's equations. In the plane wave space the curl equations are expressed as:

$$\vec{\beta}^a \times \vec{E} = \omega\mu_o\mu_r\overline{\overline{\mu}}\vec{H}, \quad \vec{\beta}^a \times \vec{H} = -\omega\varepsilon_o\varepsilon_r\overline{\overline{\varepsilon}}\vec{E}$$

where:

$$\vec{\beta}^a = \hat{x}\beta_x^i + \hat{z}\beta_z^a,$$

and:

$$\overline{\overline{\varepsilon}} = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}, \ \overline{\overline{\mu}} = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & d \end{bmatrix}$$

UNIAXIAL PML - THEORY (cont'd)

- Derive the wave equation in the Uniaxial Medium equation:

$$\vec{\beta}^{a} \times \bar{\bar{\varepsilon}}^{-1} \vec{\beta}^{a} \times \vec{H} + k^{2} \bar{\bar{\mu}} \vec{H} = 0$$

where: $k^{2} = \omega^{2} \mu_{o} \mu_{r} \varepsilon_{o} \varepsilon_{r}$.

- Expressing the cross products as matrix operators, the wave equation can be expressed more suitably in matrix form as:

$$\begin{bmatrix} k^{2} c - a^{-1} \beta_{z}^{a^{2}} & 0 & \beta_{x}^{i} \beta_{z}^{a} a^{-1} \\ 0 & k^{2} c - \beta_{z}^{a^{2}} a^{-1} - \beta_{x}^{i^{2}} b^{-1} & 0 \\ \beta_{x}^{i} \beta_{z}^{a} a^{-1} & 0 & k^{2} d - a^{-1} \beta_{x}^{i^{2}} \end{bmatrix} \begin{bmatrix} H_{x} \\ H_{y} \\ H_{z} \end{bmatrix} = 0$$

- The dispersion relationship for the uniaxial medium is derived from the determinant of the matrix operator:

$$\left(k^{2} c - a^{-1} \beta_{z}^{a^{2}} - b^{-1} \beta_{x}^{i^{2}} \right) \left(k^{2} a - c^{-1} \beta_{z}^{a^{2}} - d^{-1} \beta_{x}^{i^{2}} \right) = 0$$

- Conveniently, these solutions can be decoupled into forward and backward $\mathrm{TE_y}$ and $\mathrm{TM_y}$ modes, which satisfy the dispersion relationships:

$$k^{2} c - a^{-1} \beta_{z}^{a^{2}} - b^{-1} \beta_{x}^{i^{2}} = 0 \quad \text{for } \left(\mathrm{TE_y} \right), \ \ \& \ \ k^{2} a - c^{-1} \beta_{z}^{a^{2}} - d^{-1} \beta_{x}^{i^{2}} = 0 \text{ for } (\mathrm{TM_y})$$

UNIAXIAL PML - THEORY (cont'd)

- Initially, assume a TE$_y$ incident wave impinging on the interface:

$$\vec{H}_1 = \hat{y} H_o (1 + \Gamma e^{2j\beta_z^i z}) e^{-j\beta_x^i x - j\beta_z^i z}$$

$$\vec{E}_1 = \left(\hat{x} \tfrac{\beta_z^i}{\omega\varepsilon} (1 - \Gamma e^{2j\beta_z^i z}) - \hat{z} \tfrac{\beta_x^i}{\omega\varepsilon} (1 + \Gamma e^{2j\beta_z^i z}) \right) H_o e^{-j\beta_x^i x - j\beta_z^i z}$$

$$\vec{H}_2 = \hat{y} H_o \tau e^{-j\beta_x^i x - j\beta_z^a z}$$

$$\vec{E}_2 = \left(\hat{x} \tfrac{\beta_z^a a^{-1}}{\omega\varepsilon} - \hat{z} \tfrac{\beta_x^i b^{-1}}{\omega\varepsilon} \right) H_o \tau e^{-j\beta_x^i x - j\beta_z^a z}$$

- Enforcing field continuity at the $z = 0$ interface:

$$\Gamma = \frac{\beta_z^i - \beta_z^a a^{-1}}{\beta_z^i + \beta_z^a a^{-1}}, \quad \tau = 1 + \Gamma = \frac{2\beta_z^i}{\beta_z^i + \beta_z^a a^{-1}}$$

- The underlying objective is to determine if there exists a choice of constitutive parameters for which $\Gamma = 0$ for all angles of incidence.
- A sufficient condition is if $\beta_z^i = \beta_z^a a^{-1}$.
- Given the TE$_y$ dispersion relationship: $\beta_z^{a^2} = k^2 ca - ab^{-1} \beta_x^{i^2}$ or $\beta_z^{i^2} = k^2 ca^{-1} - \beta_x^{i^2} b^{-1} a^{-1}$
- Finally, if $c = a$ and $b = a^{-1}$, then:

$$\beta_z^{i^2} = k^2 - \beta_x^{i^2} \, !$$

- Repeating for the TM$_y$ polarization, reflectionless conditions holds if $c = a$ and $d = c^{-1}$.

UNIAXIAL ABSORBING MEDIA

- In conclusion, if:

$$\bar{\bar{\varepsilon}} = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^{-1} \end{bmatrix}, \ \bar{\bar{\mu}} = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^{-1} \end{bmatrix}$$

  a plane wave of arbitrary polarization and angle of incidence will be purely transmitted into the uniaxial medium.
- For FEM applications, the perfectly matched uniaxial medium is extremely useful if it is highly lossy such that any wave entrant upon the medium will quickly attenuate while no physical reflections will be encountered due to the interface.
- Terminating the uniaxial slab with a hard boundary such as a perfect electrical conductor (PEC), small reflections will be encountered due to the finite depth. However, if the medium is highly lossy, these reflections can be made to be extremely small.
- For a lossy uniaxial medium, one obvious choice for the constitutive parameters is $a = 1 + \frac{\sigma_z}{j\omega\varepsilon_o}$. This leads to the relative permittivity and permeability tensors:

$$\bar{\bar{\varepsilon}} = \begin{bmatrix} 1+\frac{\sigma_z}{j\omega\varepsilon_o} & 0 & 0 \\ 0 & 1+\frac{\sigma_z}{j\omega\varepsilon_o} & 0 \\ 0 & 0 & 1/(1+\frac{\sigma_z}{j\omega\varepsilon_o}) \end{bmatrix} = \bar{\bar{\mu}}$$

UNIAXIAL ABSORBING MEDIA (cont'd)

- The dispersion relationship is then expressed as:

$$k^2 = \beta_z^{a^2} / (1 + \tfrac{\sigma_z}{j\omega\varepsilon_o})^2 + \beta_x^{i^2}$$

which leads to: $\beta_z^a = \pm(1 - j\tfrac{\sigma_z}{\omega\varepsilon_o})\beta_z^i$

- Given a TE$_y$ incident wave, the field intensities in the uniaxial medium are given by:

$$\vec{H}_2 = \hat{y} H_o e^{-j\beta_x^i x - j\beta_z^i z} e^{-\alpha_z z}$$

$$\vec{E}_2 = \left(\hat{x} \frac{\beta_z^i}{\omega\varepsilon_o\varepsilon_r} - \hat{z} \frac{\beta_x^i (1 + \frac{\sigma_z}{j\omega\varepsilon_o})}{\omega\varepsilon_o\varepsilon_r} \right) H_o e^{-j\beta_x^i x - j\beta_z^i z} e^{-\alpha_z z}$$

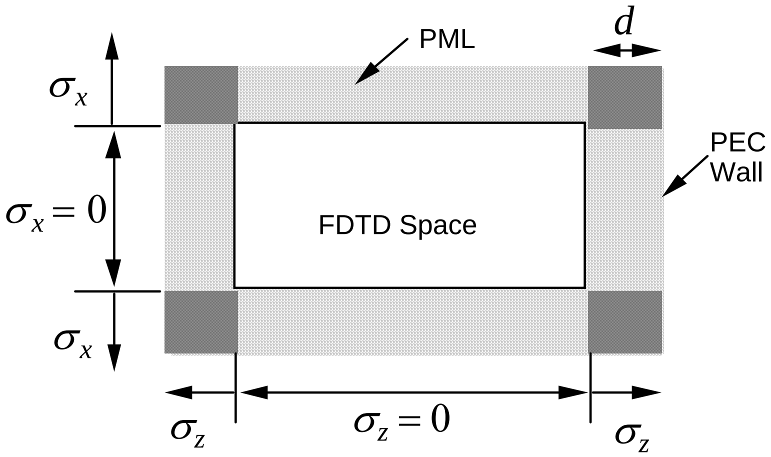
where:

$$\alpha_z = \tfrac{\sigma_z}{\omega\varepsilon_o} \beta_z^i = \sigma_z \eta_o \sqrt{\varepsilon_r} \cos\theta^i$$

- The principal advantage of the Uniaxial PML is that it is posed in a Maxwellian form. This lends to further physical insight to its properties, and has lead to the generalization of the PML to more generalized media, such as lossy, dispersive, and anisotropic media†.

† S. D. Gedney, "An Anisotropic PML Absorbing Media for FDTD Simulation of Fields in Lossy Dispersive Media," *Electromagnetics*, vol. 16, no. 3, July/August, 1996.

Corner Regions



- In General, the FEM lattice must be terminated by PML on all 6 sides. In the corner regions there are multiple interface boundaries.
- The constitutive relations in the corner regions are simply derived by matching two uniaxial media. This leads to the general expressions for Maxwell's curl equations:

$$\nabla \times \vec{H} = j\omega\varepsilon_o\varepsilon_r \overline{\overline{s}}\vec{E}, \quad \nabla \times \vec{E} = -j\omega\mu_o \overline{\overline{s}}\vec{H}$$

where:

$$\overline{\overline{s}} = \begin{bmatrix} \frac{s_y s_z}{s_x} & 0 & 0 \\ 0 & \frac{s_x s_z}{s_y} & 0 \\ 0 & 0 & \frac{s_x s_y}{s_z} \end{bmatrix}, s_x = (1 + \tfrac{\sigma_x}{j\omega\varepsilon_o}), s_y = (1 + \tfrac{\sigma_y}{j\omega\varepsilon_o}), s_z = (1 + \tfrac{\sigma_z}{j\omega\varepsilon_o})$$

- s_x, s_y, and s_z are associated with the x, y, and z-normal planes, respectively. Outside of these regions, the respective $\sigma_i = 0$.

- This can easily be derived by matching a uniaxial PML to an anisotropic medium [See Chapter 7 of *Advances in Computational Electrodynamics: The FDTD*, Artech House, 1998, A. Taflove, Ed.]

- Note that in the corner regions, the medium is no longer uniaxial. However, we refer to it as a UPML, since we are actually matching an anisotropic medium through the product with a uniaxial tensor.

IMPLEMENTATION

- The PML is implemented within the FEM formulation by treating the space as an anisotropic media

- **Galerkin Formulation:**

  Vector-Helmholtz Equation:

$$\nabla\times\overline{\overline{\mu}}_r^{-1}\nabla\times\vec{E}-k_0^2\overline{\overline{\varepsilon}}_r\vec{E}=0$$

  where, in the PML region: $\overline{\overline{\mu}}_r=\mu_r\overline{\overline{s}}$, $\overline{\overline{\varepsilon}}_r=\varepsilon_r\overline{\overline{s}}$

]

  Reaction Integral:

$$\iint_{\Omega}\left[\vec{E}^a\cdot\nabla\times\overline{\overline{\mu}}_r^{-1}\nabla\times\vec{E}-k_0^2\vec{E}^a\cdot\overline{\overline{\varepsilon}}_r\vec{E}\right]d\Omega=0$$

  Applying Green’s first-identity:

$$\iint_{\Omega}\left[\nabla\times\vec{E}^a\cdot\overline{\overline{\mu}}_r^{-1}\nabla\times\vec{E}-k_0^2\vec{E}^a\cdot\overline{\overline{\varepsilon}}_r\vec{E}\right]d\Omega-\oint_{\partial\Omega}\left[\vec{E}^a\times\overline{\overline{\mu}}_r^{-1}\nabla\times\vec{E}\right]\cdot\hat{n}dS=0$$

where, the exterior boundary is typically terminated via a PEC, or PMC, or a Sommerfeld Radiation condition.