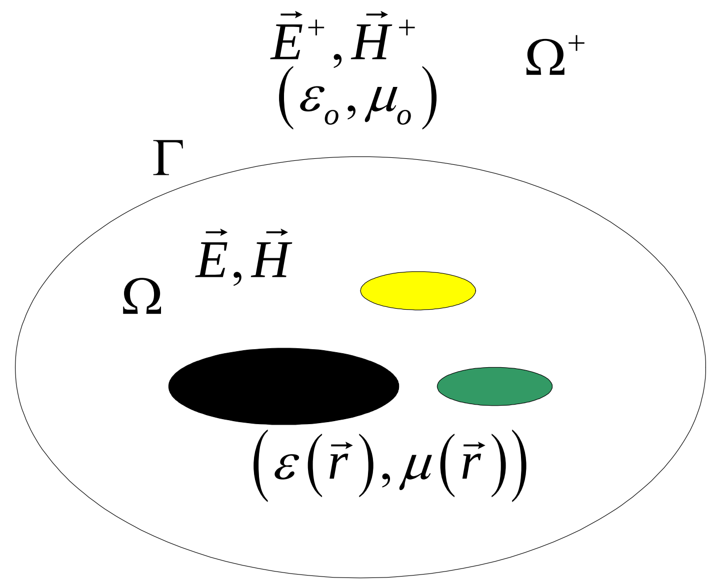
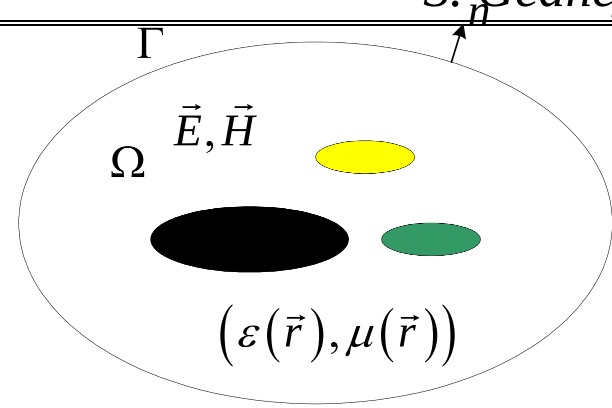
Exact Radiation Boundary Condition – Hybrid FE-BI Formulation



- We can treat the FEM space as the superposition of two domains
    - Ω – the interior domain (FEM) $(\vec{E},\vec{H})$
    - Ω^+ – the exterior domain $(\vec{E}^+,\vec{H}^+)$
- The domains are separated by the surface Γ
- If the fields in the exterior domain are known, the can be coupled to the interior fields via the boundary conditions on Γ
    - $\hat{n}\times\vec{E}\Big|_{\Gamma^-} = \hat{n}\times\vec{E}^+\Big|_{\Gamma^+}$
    - $\hat{n}\times\vec{H}\Big|_{\Gamma^-} = \hat{n}\times\vec{H}^+\Big|_{\Gamma^+}$
- In this formulation, we do not know the exterior fields, and these must also be solved. This will be done using a boundary integral (BI) formulation.
- Objectives:
    - Unique solution for $(\vec{E},\vec{H})$ in Ω and $(\vec{E}^+,\vec{H}^+)$ on Γ^+.
        - No interior resonance
    - Symmetric Formulation

The Interior FE-Problem



- The interior problem is represented by the standard FEM formulation based on the weak-form of the vector Helmholtz equation:

$$\int_{\Omega}\left(\nabla\times\vec{E}^{a}\cdot\frac{1}{\mu_r}\nabla\times\vec{E}-k_0^2\vec{E}^{a}\cdot\varepsilon_r\vec{E}\right)d\Omega-\int_{\Gamma^-}\vec{E}^{a}\times\frac{1}{\mu_r}\nabla\times\vec{E}\cdot\hat{n}ds=0$$

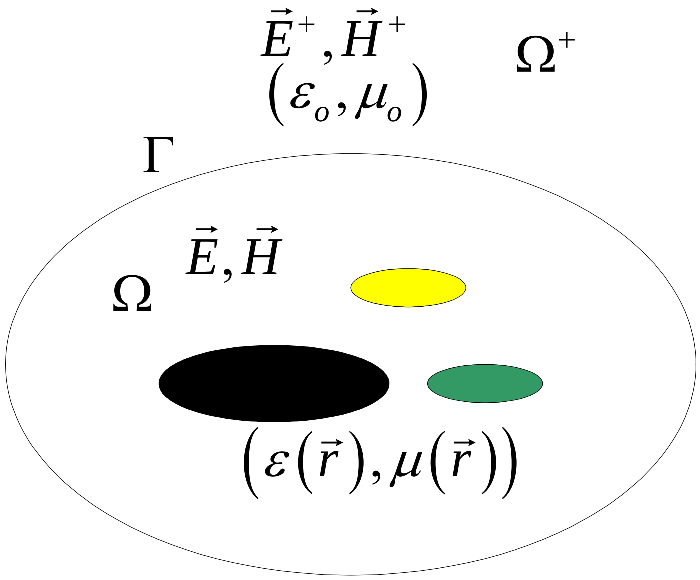
- In the boundary integral, we can note that $\because\ \nabla\times\vec{E}=-jk_0\eta_o\mu_r\vec{H}$:

$$-\int_{\Gamma^-}\vec{E}^{a}\times\frac{1}{\mu_r}\nabla\times\vec{E}\cdot\hat{n}ds=jk_o\eta_o\int_{\Gamma^-}\vec{E}^{a}\times\vec{H}\cdot\hat{n}ds=-jk_0\eta_o\int_{\Gamma^-}\vec{E}^{a}\cdot\hat{n}\times\vec{H}ds$$

- Therefore, this can be expressed as:

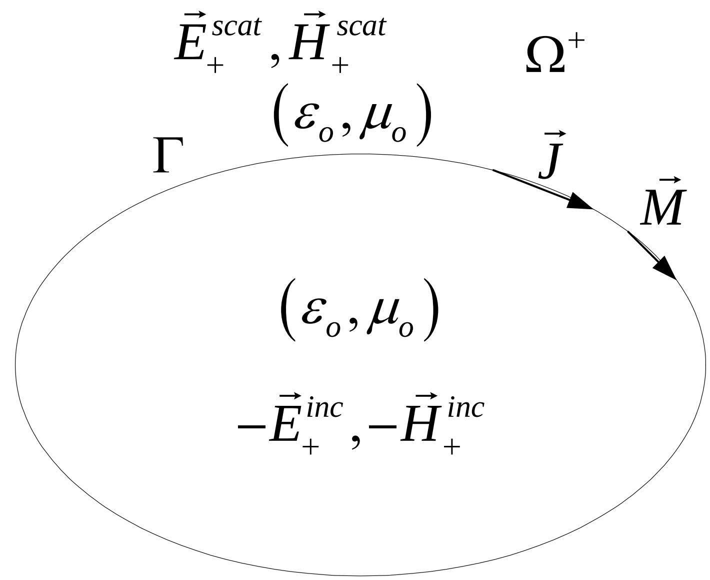
$$\int_{\Omega}\left(\nabla\times\vec{E}^{a}\cdot\frac{1}{\mu_r}\nabla\times\vec{E}-k_0^2\vec{E}^{a}\cdot\varepsilon_r\vec{E}\right)d\Omega-jk_0\eta_o\int_{\Gamma^-}\vec{E}^{a}\cdot\hat{n}\times\vec{H}ds=0 \qquad (1)$$

The Exterior Problem



- In the exterior region, the electric and magnetic fields can be distinguished as being a superposition of the *incident* and *scattered* fields:
  - $\vec{E}^+ = \vec{E}_+^{inc} + \vec{E}_+^{scat}$
  - $\vec{H}^+ = \vec{H}_+^{inc} + \vec{H}_+^{scat}$
    - *Incident fields* are the fields excited by an impressed source in Ω^+ in the absence of any inhomogeneity inside Ω.
    - *Scattered fields* are the perturbation of the field due to any inhomogeneity in Ω.
    - Therefore, the *total* field is the superposition of the incident and scattered fields.
- It is assumed that we know the incident field (typically in a closed form). For example, it can be a uniform plane wave.
- However, we do *not* know the scattered field, since this is due to whatever is inside Ω. We need to somehow calculate these fields

The Exterior BI-Problem



- The exterior scattered field can be calculated using *Equivalence Principals*.
- Pose the equivalent surface current densities:
  - $\vec{J} = \hat{n} \times \vec{H}^+ \big|_{\Gamma^+}$
  - $\vec{M} = \vec{E}^+ \times \hat{n} \big|_{\Gamma^+}$
- These currents reside on Γ, and radiate in a homogeneous free space.
  - Radiate $\vec{E}_+^{scat}, \vec{H}_+^{scat}$ in Ω^+
  - Radiate $\vec{E}_+^{inc}, \vec{H}_+^{inc}$ in Ω
- The scattered fields are determined via Vector Potential Theory:

$$\vec{E}^{scat}(\vec{r}) = -jk_o\eta_o\vec{A}(\vec{r}) + \frac{\eta_o}{jk_o}\nabla\nabla\cdot\vec{A}(\vec{r}) - \nabla\times\vec{F}(\vec{r}) \quad (2)$$

$$\vec{H}^{scat}(\vec{r}) = -\frac{jk_o}{\eta_o}\vec{F}(\vec{r}) + \frac{1}{jk_o\eta_o}\nabla\nabla\cdot\vec{F}(\vec{r}) + \nabla\times\vec{A}(\vec{r}) \quad (3)$$

where,

$$\vec{A}(\vec{r}) = \int_{\Gamma} \vec{J}(\vec{r}\,') \frac{e^{-jk_o|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} ds, \quad \vec{F}(\vec{r}) = \int_{\Gamma} \vec{M}(\vec{r}\,') \frac{e^{-jk_o|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} ds \tag{4}$$

- Next, we will pose the boundary conditions on Γ^+

$$\vec{J} = \hat{n} \times \vec{H}^+ \Big|_{\Gamma^+} = \hat{n} \times \left(\vec{H}_+^{inc} + \vec{H}_+^{scat} \right) \Big|_{\Gamma^+} \tag{5}$$

$$\vec{M} = \vec{E}^+ \times \hat{n} \Big|_{\Gamma^+} = \left(\vec{E}_+^{inc} + \vec{E}_+^{scat} \right) \times \hat{n} \Big|_{\Gamma^+} \tag{6}$$

- Then, from (2) and (3)

$$\vec{J} = \hat{n} \times \vec{H}_+^{inc} \Big|_{\Gamma^+} + \hat{n} \times \left(-\frac{jk_o}{\eta_o} \vec{F}(\vec{r}) + \frac{1}{jk_o\eta_o} \nabla\nabla \cdot \vec{F}(\vec{r}) + \nabla \times \vec{A}(\vec{r}) \right) \Bigg|_{\vec{r} \in \Gamma^+} \tag{7}$$

$$\vec{E}^+ \times \hat{n} \Big|_{\Gamma^+} = \vec{E}_+^{inc} \times \hat{n} \Big|_{\Gamma^+} + \left(-jk_o\eta_o \vec{A}(\vec{r}) + \frac{\eta_o}{jk_o} \nabla\nabla \cdot \vec{A}(\vec{r}) - \nabla \times \vec{F}(\vec{r}) \right) \times \hat{n} \Bigg|_{\Gamma} \tag{8}$$

- Next, as in (1), we need to pose a reaction integral. To this end, we will test (7) with $\vec{E}$, and we will test (8) with $\vec{J}$. This leads to:

$$\begin{aligned} \int_{\Gamma} \vec{E}^a \cdot \vec{J} ds = & \int_{\Gamma} \vec{E}^a \cdot \hat{n} \times \vec{H}_+^{inc} ds + \\ & \int_{\Gamma} \vec{E}^a \cdot \hat{n} \times \left(-\frac{jk_o}{\eta_o} \vec{F}(\vec{r}) + \frac{1}{jk_o\eta_o} \nabla\nabla \cdot \vec{F}(\vec{r}) + \nabla \times \vec{A}(\vec{r}) \right) ds \end{aligned} \tag{9}$$

- Permuting the triple-scalar product, this can be expressed more appropriately as:

$$\int_{\Gamma} \vec{E}^a \cdot \vec{J} ds = \int_{\Gamma} \vec{E}^a \times \hat{n} \cdot \vec{H}_+^{inc} ds - \frac{jk_o}{\eta_o} \int_{\Gamma} \vec{E}^a \times \hat{n} \cdot \vec{F}(\vec{r}) ds + \frac{1}{jk_o\eta_o} \int_{\Gamma} \vec{E}^a \times \hat{n} \cdot \nabla\nabla \cdot \vec{F}(\vec{r}) ds + \int_{\Gamma^+} \vec{E}^a \times \hat{n} \cdot \nabla \times \vec{A}(\vec{r}) ds \quad (10)$$

- From (8), we take advantage of the identity: $\hat{n} \times \vec{E} \times \hat{n} = \vec{E}_t$, where $\vec{E}_t$ is tangential to Γ. Then, performing the inner product with $\vec{J}$ (which is tangential to Γ), leads to:

$$\int_{\Gamma} \vec{J}^a \cdot \vec{E}^+ ds = \int_{\Gamma} \vec{J}^a \cdot \vec{E}_+^{inc} ds - jk_o\eta_o \int_{\Gamma} \vec{J}^a \cdot \vec{A}(\vec{r}) ds + \frac{\eta_o}{jk_o} \int_{\Gamma} \vec{J}^a \cdot \nabla\nabla \cdot \vec{A}(\vec{r}) ds - \int_{\Gamma^+} \vec{J}^a \cdot \nabla \times \vec{F}(\vec{r}) ds \quad (11)$$

- Finally, in an attempt to add symmetry to (10) and (11), we will substitute $\vec{J} = \underset{\sim}{J} / jk_o\eta_o$, which leads to: $\vec{A} = \underset{\sim}{A} / jk_o\eta_o$. Furthermore, (10) is multiplied by $jk_o\eta_o$. This leads to:

Exterior Integral Equations

$$\int_{\Gamma} \vec{E}^a \cdot \underset{\sim}{J} ds = jk_o\eta_o \int_{\Gamma} \vec{E}^a \times \hat{n} \cdot \vec{H}_+^{inc} ds + k_o^2 \int_{\Gamma} \vec{E}^a \times \hat{n} \cdot \vec{F}(\vec{r}) ds + \int_{\Gamma} \vec{E}^a \times \hat{n} \cdot \nabla\nabla \cdot \vec{F}(\vec{r}) ds + \int_{\Gamma^+} \vec{E}^a \times \hat{n} \cdot \nabla \times \underset{\sim}{A}(\vec{r}) ds \tag{12}$$

and

$$\int_{\Gamma} \vec{J}^a \cdot \vec{E}^+ ds = \int_{\Gamma} \vec{J}^a \cdot \vec{E}_+^{inc} ds - \int_{\Gamma} \vec{J}^a \cdot \underset{\sim}{A}(\vec{r}) ds - \frac{1}{k_o^2} \int_{\Gamma} \vec{J}^a \cdot \nabla\nabla \cdot \underset{\sim}{A}(\vec{r}) ds - \int_{\Gamma^+} \vec{J}^a \cdot \nabla \times \vec{F}(\vec{r}) ds \tag{13}$$

- Finally, it is recognized that the $\nabla \times \vec{A}$ and $\nabla \times \vec{F}$ require principal value integrals. The reason for this is two fold. Initially, the field radiated through these terms are discontinuous across Γ. This is why we are careful in labeling these integrals as being over Γ^+. Secondly, these terms are hyper-singular, and must be evaluated via the principal value integral.
- The principal value integral can be done analytically. This leads to:

Exterior Integral Equations

$$\frac{1}{2}\int_{\Gamma}\vec{E}^{a}\cdot\underset{\sim}{\vec{J}}ds = jk_o\eta_o\int_{\Gamma}\vec{E}^{a}\times\hat{n}\cdot\vec{H}_{+}^{inc}ds + k_o^2\int_{\Gamma}\vec{E}^{a}\times\hat{n}\cdot\vec{F}(\vec{r})ds + \int_{\Gamma}\vec{E}^{a}\times\hat{n}\cdot\nabla\nabla\cdot\vec{F}(\vec{r})ds + ⨍_{\Gamma}\vec{E}^{a}\times\hat{n}\cdot\nabla\times\underset{\sim}{\vec{A}}(\vec{r})ds \tag{14}$$

and

$$\frac{1}{2}\int_{\Gamma}\vec{J}^{a}\cdot\vec{E}^{+}ds = \int_{\Gamma}\vec{J}^{a}\cdot\vec{E}_{+}^{inc}ds - \int_{\Gamma}\vec{J}^{a}\cdot\underset{\sim}{\vec{A}}(\vec{r})ds - \frac{1}{k_o^2}\int_{\Gamma}\vec{J}^{a}\cdot\nabla\nabla\cdot\underset{\sim}{\vec{A}}(\vec{r})ds - ⨍_{\Gamma}\vec{J}^{a}\cdot\nabla\times\vec{F}(\vec{r})ds \tag{15}$$

Coupling the Exterior and Interior Regions

- At this point we have posed separate formulations for the interior and the exterior region problems, which are summarized by (1), (12), and (13)
- The interior and exterior problems can be combined by enforcing the continuity of the tangential fields:

$$\hat{n}\times\vec{H}^{+}\Big|_{\Gamma^{+}} = \hat{n}\times\vec{H}\Big|_{\Gamma^{-}} \tag{16}$$

$$\vec{E}^{+}\times\hat{n}\Big|_{\Gamma^{+}} = \vec{E}\times\hat{n}\Big|_{\Gamma^{-}} \tag{17}$$

- These can also be related to the equivalence principal:

$$\hat{n}\times\vec{H}^{+}\Big|_{\Gamma^{+}} = \hat{n}\times\vec{H}\Big|_{\Gamma^{-}} = \vec{J} \tag{18}$$

$$\vec{E}^{+}\times\hat{n}\Big|_{\Gamma^{+}} = \vec{E}\times\hat{n}\Big|_{\Gamma^{-}} = \vec{M} \tag{19}$$

where, $\hat{n}$ is directed into Ω^{+}

- From (18), we can then re-write (1) as:

$$\boxed{\int_{\Gamma^{-}} \vec{E}^{a}\cdot\underset{\sim}{\vec{J}}ds = \int_{\Omega}\left(\nabla\times\vec{E}^{a}\cdot\frac{1}{\mu_r}\nabla\times\vec{E} - k_0^2\vec{E}^{a}\cdot\varepsilon_r\vec{E}\right)d\Omega} \tag{20}$$

where, $\underset{\sim}{\vec{J}} = jk_o\eta_o\vec{J}$

Coupling the Exterior and Interior Regions (cont'd)

- Comparing (20) with (14), it is observed that these terms can be combined.
  - o We would like to do this in a manner that preserves symmetry of the global operator.
- We can write (20) as:[1]

$$\frac{1}{2}\int_{\Gamma^-} \vec{E}^a \cdot \underset{\sim}{\vec{J}} ds + \frac{1}{2}\int_{\Gamma^-} \vec{E}^a \cdot \underset{\sim}{\vec{J}} ds = \int_{\Omega}\left(\nabla\times\vec{E}^a \cdot \frac{1}{\mu_r}\nabla\times\vec{E} - k_0^2 \vec{E}^a \cdot \varepsilon_r \vec{E}\right) d\Omega \tag{21}$$

- Then, substituting in (14), leads to:

$$\begin{aligned}&\frac{1}{2}\int_{\Gamma^-} \vec{E}^a \cdot \underset{\sim}{\vec{J}} ds + jk_o\eta_o\int_{\Gamma} \vec{E}^a \times \hat{n}\cdot \vec{H}_+^{inc} ds + k_o^2\int_{\Gamma}\vec{E}^a\times\hat{n}\cdot\vec{F}\left(\vec{r}\right)ds + \\ &\int_{\Gamma}\vec{E}^a\times\hat{n}\cdot\nabla\nabla\cdot\vec{F}\left(\vec{r}\right)ds + \fint_{\Gamma}\vec{E}^a\times\hat{n}\cdot\nabla\times\underset{\sim}{\vec{A}}\left(\vec{r}\right)ds \\ &= \int_{\Omega}\left(\nabla\times\vec{E}^a\cdot\frac{1}{\mu_r}\nabla\times\vec{E} - k_0^2\vec{E}^a\cdot\varepsilon_r\vec{E}\right)d\Omega\end{aligned} \tag{22}$$

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[1] Vouvakis, Lee, Zhao, & J-F Lee, IEEE Trans. Ant. Prop., vol 52, pp. 3060-3070, Nov. 2004.

Coupling the Exterior and Interior Regions (cont'd)

- We can re-arrange (22) as:

$$\begin{aligned}
&\frac{1}{2}\int_{\Gamma^-} \vec{E}^a \cdot \underset{\sim}{J} ds = \\
&\quad \int_\Omega \left(\nabla \times \vec{E}^a \cdot \frac{1}{\mu_r} \nabla \times \vec{E} - k_0^2 \vec{E}^a \cdot \varepsilon_r \vec{E} \right) d\Omega - \\
&\quad jk_o \eta_o \int_\Gamma \vec{E}^a \times \hat{n} \cdot \vec{H}_+^{inc} ds - k_o^2 \int_\Gamma \vec{E}^a \times \hat{n} \cdot \vec{F}(\vec{r}) ds - \\
&\quad \int_\Gamma \vec{E}^a \times \hat{n} \cdot \nabla\nabla \cdot \vec{F}(\vec{r}) ds - \fint_\Gamma \vec{E}^a \times \hat{n} \cdot \nabla \times \underset{\sim}{A}(\vec{r}) ds
\end{aligned} \tag{23}$$

- This is finally combined with (15) and (20), leading to a complete description of the coupled FE-BI formulation

Summary of the FE-BI formulation

- Interior Formulation:

$$\int_{\Omega}\left(\nabla\times\vec{E}^{a}\cdot\frac{1}{\mu_r}\nabla\times\vec{E}-k_0^2\vec{E}^{a}\cdot\varepsilon_r\vec{E}\right)d\Omega-\int_{\Gamma^-}\vec{E}^{a}\cdot\underset{\sim}{\vec{J}}ds=0 \quad (24)$$

- Coupled Formulation

$$\begin{aligned}&\int_{\Omega}\left(\nabla\times\vec{E}^{a}\cdot\frac{1}{\mu_r}\nabla\times\vec{E}-k_0^2\vec{E}^{a}\cdot\varepsilon_r\vec{E}\right)d\Omega-\frac{1}{2}\int_{\Gamma^-}\vec{E}^{a}\cdot\underset{\sim}{\vec{J}}ds\\&-k_o^2\int_{\Gamma}\vec{E}^{a}\times\hat{n}\cdot\vec{F}(\vec{r})ds-\int_{\Gamma}\vec{E}^{a}\times\hat{n}\cdot\nabla\nabla\cdot\vec{F}(\vec{r})ds\\&-\int_{\Gamma}\vec{E}^{a}\times\hat{n}\cdot\nabla\times\underset{\sim}{\vec{A}}(\vec{r})ds \quad = jk_o\eta_o\int_{\Gamma}\vec{E}^{a}\times\hat{n}\cdot\vec{H}_{+}^{inc}ds\end{aligned} \quad (25)$$

- Exterior Formulation

$$\begin{aligned}&-\frac{1}{2}\int_{\Gamma}\vec{J}^{a}\cdot\vec{E}^{+}ds-\int_{\Gamma}\vec{J}^{a}\cdot\underset{\sim}{\vec{A}}(\vec{r})ds-\frac{1}{k_o^2}\int_{\Gamma}\vec{J}^{a}\cdot\nabla\nabla\cdot\underset{\sim}{\vec{A}}(\vec{r})ds\\&-\int_{\Gamma}\vec{J}^{a}\cdot\nabla\times\vec{F}(\vec{r})ds \quad =-\int_{\Gamma}\vec{J}^{a}\cdot\vec{E}_{+}^{inc}ds\end{aligned} \quad (26)$$

Discretization of the FE-BI formulation

- The domain Ω is discretized via 3D fitted polyhedron (e.g., tetrahedron)
- The surface Γ is then the surface mesh resulting from the 3D polyhedral mesh (e.g., triangles).
    - $\vec{E}$ is expanded using H(p)-curl conforming vector basis functions in Ω
    - The support of $\vec{E}$ on Γ is an H(p)-curl surface basis
        - These basis are supported by the sub-topologies on Γ (i.e., faces and edges)
    - $\vec{J}$ is expanded using H(p)-divergence conforming vector basis functions on Γ.
- Galerkin formulation:
    - $\vec{E}^a$ and $\vec{E}$ are chosen to span the same H(p)-curl conforming function space in Ω and on Γ
    - $\vec{J}^a$ and $\vec{J}$ are chosen to span the same H(p)-div conforming function space on Γ
- The magnetic current density on Γ is expanded as: $\vec{M} = \vec{E} \times \hat{n}\big|_{\Gamma}$
    - $\vec{E} \times \hat{n}$ transforms the H(p)-curl conforming function into the H(p)-divergence conforming function space on Γ.
        - Identical to the space used to expand *J*

Deriving the System Matrix

- Following the method of weighted residuals, the electric field and the electric surface current density are expanded via the function spaces as discussed above. The functions are weighted by unknown constant coefficients
  - Let the vector of coefficients representing the fields interior to Ω be $\overline{e}_\Omega$
  - Let the vector of coefficients representing the fields on Γ be $\overline{e}_\Gamma$
  - Let the vector of coefficients weighting the surface current be $\overline{j}$
  - Let $\vec{M} = \vec{E} \times \hat{n}$ on Γ
- This leads to the linear system of equations:

$$\begin{pmatrix} \overline{\overline{K}}_{\Omega,\Omega} & \overline{\overline{K}}_{\Omega,\Gamma} & 0 \\ \overline{\overline{K}}_{\Omega,\Gamma}^T & \overline{\overline{K}}_{\Gamma,\Gamma} + \overline{\overline{L}}_{m,m} & \overline{\overline{D}} + \overline{\overline{Q}} \\ 0 & \overline{\overline{D}}^T + \overline{\overline{Q}}^T & \overline{\overline{L}}_{e,e} \end{pmatrix} \begin{pmatrix} \overline{e}_\Omega \\ \overline{e}_\Gamma \\ \overline{j} \end{pmatrix} = \begin{pmatrix} 0 \\ b_h \\ b_e \end{pmatrix} \quad (27)$$

- where,

$$\left(K_{\Omega,\Omega}\right)_{i,j} = \int_\Omega \left(\nabla \times \vec{w}_i \cdot \frac{1}{\mu_r} \nabla \times \vec{w}_j - k_0^2 \vec{w}_i \cdot \varepsilon_r \vec{w}_j \right) d\Omega \quad (28)$$

and $K_{\Omega,\Gamma}$ is similarly defined with $\vec{w}_j$ supported on Γ. (similar for $K_{\Gamma,\Gamma}$)

$$\left(L_{m,m}\right)_{i,j} = -k_o^2 \int_\Gamma \vec{w}_i \times \hat{n} \cdot \vec{F}_j\left(\vec{r}\right) ds - \int_\Gamma \vec{w}_i \cdot \nabla\nabla \cdot \vec{F}_j\left(\vec{r}\right) ds, \text{ and } \vec{M}_j = \vec{w}_j \times \hat{n} \quad (29)$$

$$\left(D\right)_{i,j} = -\frac{1}{2} \int_{\Gamma^-} \vec{w}_i \cdot \vec{\Lambda}_j ds \quad (30)$$

where, $\vec{\Lambda}_j = \vec{w}_j \times \hat{n}$

$$\left(Q\right)_{i,j} = -\int_\Gamma \vec{w}_i \times \hat{n} \cdot \nabla \times \underset{\sim}{\vec{A}}_j ds, \text{ and } \underset{\sim}{\vec{J}}_j = \vec{\Lambda}_j = \vec{w}_j \times \hat{n} \quad (31)$$

and

$$\left(L_{e,e}\right)_{i,j} = k_o^2 \left(L_{m,m}\right)_{i,j} \quad (32)$$