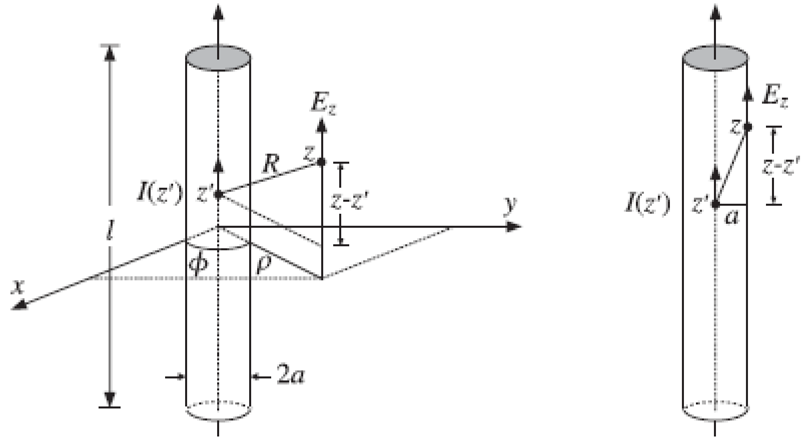
Thin Wire Antenna



- Consider a straight thin-wire antenna excited at the center by an arbitrary source
- Objective:
  - o Compute the fields scattered (or radiated) by the antenna
- Solution
  - o EFIE formulation – assume wire is perfectly conducting
  - o Use the thin-wire kernel approximation
  - o Use a method of moment discretization based on triangular basis and pulse test functions

The EFIE

- The electric field integral equation (EFIE) was derived by enforcing the physical boundary condition of the tangential electric field on the surface of the PEC:

$$\hat{n}\times\vec{E}^{inc}\Big|_C = -\hat{n}\times\vec{E}^{scat}\Big|_C$$

- The scattered field is computed as:

$$\vec{E}^{scat}(\vec{r}) = -jk_o\eta_o\vec{A}(\vec{r}) + \frac{\eta_o}{jk_o}\nabla\nabla\cdot\vec{A}(\vec{r})$$

  - where $k_o\eta_o = \omega\mu_o$, $\eta_o / k_o = 1/\omega\varepsilon_o$, and

$$\vec{A}(\vec{r}) = \int_C \frac{e^{-jk|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}\vec{J}_s(\vec{r}')\,d\ell'$$

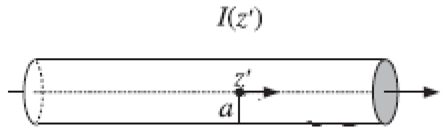
- The boundary condition can be enforced by projecting the total field on the tangential vector directly. Thus, the EFIE can be written as:

$$\hat{t}\cdot\vec{E}^{inc}(\vec{r}) = jk_o\eta_o\hat{t}\cdot\vec{A}(\vec{r}) - \frac{\eta_o}{jk_o}\hat{t}\cdot\nabla\nabla\cdot\vec{A}(\vec{r}),\quad \vec{r}\in C$$

- Or, in terms of the scalar potential ($\Phi_e = -\frac{1}{j\omega\varepsilon_o}\nabla\cdot\vec{A}$):

$$\hat{t}\cdot\vec{E}^{inc}(\vec{r}) = jk_o\eta_o\hat{t}\cdot\vec{A}(\vec{r}) + \hat{t}\cdot\nabla\Phi_e(\vec{r}),\quad \vec{r}\in C$$

The Thin Wire Approximation



- Assumptions
  - Assume that the wire radius $a \ll \lambda$, and $a \ll L$, the length of the antenna.
  - Assume that the current density of the wire is purely axial (i.e., $J_\phi = 0$)
  - Assume that the axial current is uniformly distributed about the circumference of the wire
  - Let $I(t)$ be the net axial current
    - $I(t) = \int_0^{2\pi} J_t(\phi)\, a\, d\phi$
  - Thus:
    - $J_t(t) = \dfrac{I(t)}{2\pi a}$

The Method of Weighted Residuals Solution

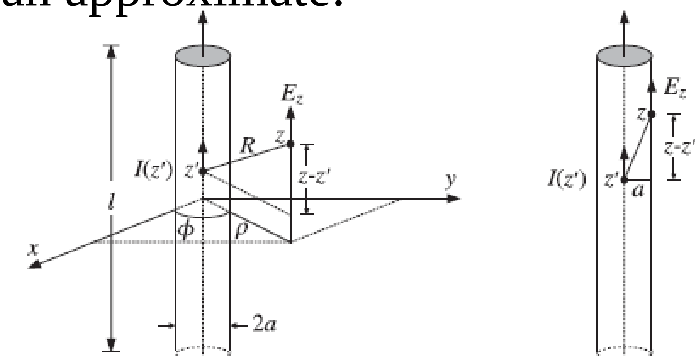
$$\int_S \vec{T}(\vec{r}) \cdot \vec{E}^{inc}(\vec{r})\, ds = jk_o\eta_o \int_S \vec{T}(\vec{r}) \cdot \int_S \vec{J}_s(\vec{r}\,') \frac{e^{-jkR}}{4\pi R} ds' + \int_S \vec{T}(\vec{r}) \cdot \nabla\Phi_e(\vec{r})\, ds$$

- o where $\Phi_e = -\dfrac{1}{j\omega\varepsilon_o} \displaystyle\int_S \nabla' \cdot \vec{J}_s(\vec{r}\,') \frac{e^{-jkR}}{4\pi R} ds'$

- Due to thin wire approximation, $\vec{J}_s(\vec{r}\,')$ is a function of the axial direction t only
- Basis functions interpolating the current must be sufficiently smooth due to $\nabla' \cdot \vec{J}_s$
  - o $\nabla' \cdot \vec{J}_s = \dfrac{\partial}{\partial t'} J_t(t')$
    - ▪ Choose triangular basis functions (linear approximation)
- Similarly, the test functions will also only be a function of the axial direction
- Test functions $\vec{T}(\vec{r})$ must be sufficiently smooth
    - ▪ integration by parts of $\vec{T}(\vec{r}) \cdot \nabla\Phi_e(\vec{r})$ leads to dervatives on T
  - o Choose pulse test functions.

The Thin-Wire Kernel

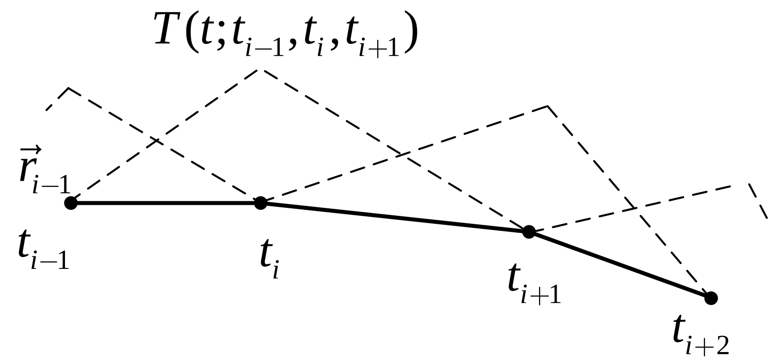
- The EFIE requires integrals of the form:
  - $\int_S J_t\left(r'(t)\right)\dfrac{e^{-jkR}}{4\pi R}ds'$
    - where $R = |\vec{r} - r'|$ and S is a single wire segment.
- Assuming the wire is thin, and $J_t(\vec{r}') = \dfrac{I(t')}{2\pi a}$ one can approximate:
  - $\int_S J_t\left(r(t')\right)\dfrac{e^{-jkR}}{4\pi R}ds' \approx \int_t I(t')\dfrac{e^{-jkR_{TW}}}{4\pi R_{TW}}dt'$
    - where:
      - $R_{TW} = \left|\left(\vec{r}(t) + \hat{\rho}a\right) - \vec{r}(t')\right|$
      - and $\vec{r}(t)$, $\vec{r}(t')$ lie on the ***wire axis***, and $\hat{\rho}$ is normal to the wire axis at t and and at angle $\phi = 0$ (relative to the local wire geometry).
    - If the wire is a straight wire (no bends),
      - $R_{TW} = \sqrt{\left(t - t'\right)^2 + a^2}$



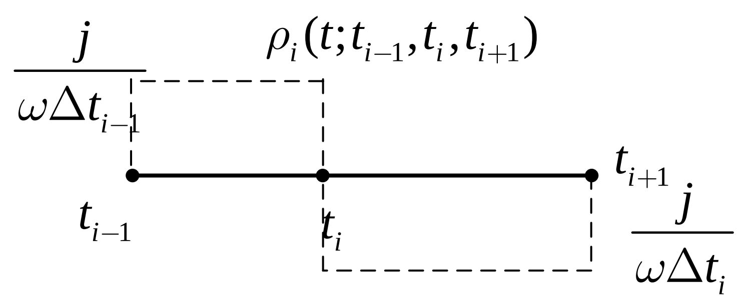
- The EFIE can be re-written as:
  - $\int_t \vec{T}(t)\cdot\vec{E}^{inc}\left(\vec{r}(t)\right)dt = jk_o\eta_o\int_t \vec{T}(t)\cdot\int_{t'}\hat{t}'I(t')\dfrac{e^{-jkR_{TW}}}{4\pi R_{TW}}dt'dt + \int_t \vec{T}(t)\cdot\nabla\Phi_e\left(\vec{r}(t)\right)dt$

Discretization

- Linear segment approximation of the thin-wire axis
- Triangular basis function expansion
  - o $I(t) \approx \sum_{i=1}^{N} \alpha_i T(t; t_{i-1}, t_i, t_{i+1})$
    - ▪ $T(t; t_{i-1}, t_i, t_{i+1}) = \begin{cases} \dfrac{t - t_{i-1}}{\Delta t_{i-1}}, & t_{i-1} \le t \le t_i \\ \dfrac{t_{i+1} - t}{\Delta t_i}, & t_i \le t \le t_{i+1} \end{cases}$

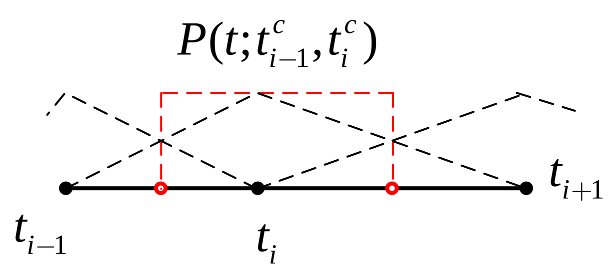


- Charge density:
  - o $\rho = \dfrac{-1}{j\omega} \dfrac{\partial}{\partial t} I(t) \approx \dfrac{-1}{j\omega} \sum_{i=1}^{N} \alpha_i \dfrac{\partial}{\partial t} T(t; t_{i-1}, t_i, t_{i+1}) = \dfrac{j}{\omega} \begin{cases} \dfrac{1}{\Delta t_{i-1}}, & t_{i-1} \le t \le t_i \\ \dfrac{-1}{\Delta t_i}, & t_i \le t \le t_{i+1} \end{cases}$
    - ▪ Charge doublet



- The charge density can also be written as a superposition of pulse functions:
    - $\rho \approx \frac{j}{\omega}\sum_{i=1}^{N}\alpha_i\left[\frac{1}{\Delta_{i-1}}P\left(t;t_{i-1},t_i\right)-\frac{1}{\Delta_i}P\left(t;t_i,t_{i+1}\right)\right]$
        - Pulse function: $P\left(t;t_i,t_{i+1}\right)=\begin{cases}1, & t_i<t<t_{i+1}\\ 0, & \text{else}\end{cases}$

- Pulse test functions
    - There is not sufficient smoothness to use delta-test functions
    - Choose pulse functions
        - $\vec{T}_i=\hat{t}P\left(t;t_{i-1}^c,t_i^c\right)$
        - Chosen to be centered about the edge nodes



Evaluating the System Matrix – The Charge Term

- The electric scalar potential is expressed as:
  - $\Phi_e(\vec{r}) = \frac{1}{\varepsilon_o}\int_t \rho(t')\frac{e^{-jkR_{TW}}}{4\pi R_{TW}}dt'$
- In the discrete space $\rho \approx \frac{j}{\omega}\sum_{i=1}^{N}\alpha_i\left[\frac{1}{\Delta_{i-1}}P(t;t_{i-1},t_i) - \frac{1}{\Delta_i}P(t;t_i,t_{i+1})\right]$. Therefore,
  - $\Phi_e(\vec{r}) = \frac{j\eta_o}{k_o}\sum_{i=1}^{N}\alpha_i\left[\frac{1}{\Delta t_{i-1}}\int_{t_{i-1}}^{t_i}\frac{e^{-jkR_{TW}}}{4\pi R_{TW}}dt' - \frac{1}{\Delta t_i}\int_{t_i}^{t_{i+1}}\frac{e^{-jkR_{TW}}}{4\pi R_{TW}}dt'\right]$
- Next, consider the outer integral due to the inner product:
  - $\int_t T_j(\vec{r})\hat{t}\cdot\nabla\Phi_e(t)dt = \int_{t_{j-1}^c}^{t_j^c}\hat{t}\cdot\nabla\Phi_e(t)dt = \int_{t_{j-1}^c}^{t_j^c}\frac{d}{dt}\Phi_e(t)dt = \Phi_e(t_j^c) - \Phi_e(t_{j-1}^c)$
- From above, this then becomes (where $R_j^c = |\vec{r}(t_j^c) + \hat{\rho}a - \vec{r}(t')|$):
  - $$\int_C \vec{T}_j(\vec{r})\cdot\nabla\Phi_e(\vec{r})dt = \frac{j\eta_o}{k_o}\sum_{i=1}^{N}\alpha_i\left[\begin{array}{l}\frac{1}{\Delta t_{i-1}}\int_{t_{i-1}}^{t_i}\frac{e^{-jkR_j^c}}{4\pi R_j}dt' - \frac{1}{\Delta t_i}\int_{t_i}^{t_{i+1}}\frac{e^{-jkR_j^c}}{4\pi R_j}dt' \\ -\frac{1}{\Delta t_{i-1}}\int_{t_{i-1}}^{t_i}\frac{e^{-jkR_{j-1}^c}}{4\pi R_{j-1}}dt' + \frac{1}{\Delta t_i}\int_{t_i}^{t_{i+1}}\frac{e^{-jkR_{j-1}^c}}{4\pi R_{j-1}}dt'\end{array}\right]$$

Evaluating the Vector Potential Term:

- Next, consider the evaluation of the vector potential term:
  - $$jk_o\eta_o \int_{t_{j-1/2}}^{t_{j+1/2}} \hat{t}\cdot \int_{t_{i-1}}^{t_{i+1}} \hat{t}'T\left(t';t_{i-1},t_i,t_{i+1}\right)\frac{e^{-jkR_{TW}}}{4\pi R_{TW}}dt'dt$$
- The inner integral is approximated by equating the moments:
  - $$\int_{t_{i-1}}^{t_{i+1}} \hat{t}'T\left(t';t_{i-1},t_i,t_{i+1}\right)\frac{e^{-jkR_{TW}}}{4\pi R_{TW}}dt' \approx \int_{t_{i-1/2}}^{t_{i+1/2}} \hat{t}'P\left(t';t_{i-1/2},t_{i+1/2}\right)\frac{e^{-jkR_{TW}}}{4\pi R_{TW}}dt'$$
- The inner integral is approximated via a mid-point rule, leading to:
  - $$jk_o\eta_o \int_{t_{j-1/2}}^{t_{j+1/2}} \hat{t}\cdot \int_{t_{i-1}}^{t_{i+1}} \hat{t}'T\left(t';t_{i-1},t_i,t_{i+1}\right)\frac{e^{-jkR_{TW}}}{4\pi R_{TW}}dt'dt \approx$$
    $$jk_o\eta_o\left[\frac{\Delta t_{j-1}}{2}\hat{t}_{j-1}+\frac{\Delta t_j}{2}\hat{t}_j\right]\cdot\int_{t_{i-1/2}}^{t_{i+1/2}}\hat{t}'\frac{e^{-jkR_j}}{4\pi R_j}dt'$$
    - where $R_j = |\vec{r}\left(t_j\right)+\hat{\rho}a-\vec{r}\left(t'\right)|$, Δt_j is the length of the *j*-th edge, and $\hat{t}_j$ is the tangent of the *j*-th edge.

The Discrete Thin-Wire EFIE

- The forcing vector is finally computed as:
  - $\int_{t_{j-1/2}}^{t_{j+1/2}} \hat{t}\cdot\vec{E}^{inc}\left(\vec{r}\left(t\right)\right)dt \approx \left[\frac{\Delta t_{j-1}}{2}\hat{t}_{j-1}+\frac{\Delta t_{j}}{2}\hat{t}_{j}\right]\cdot\vec{E}^{inc}\left(\vec{r}\left(t_j\right)\right)$
- Putting all of this together, the discrete thin-wire EFIE can be written as:

$$\left[\frac{\Delta t_{j-1}}{2}\hat{t}_{j-1}+\frac{\Delta t_{j}}{2}\hat{t}_{j}\right]\cdot\vec{E}^{inc}\left(\vec{r}\left(t_j\right)\right)=$$

$$\sum_{i=1}^{N}\alpha_i\left\{jk_o\eta_o\left[\frac{\Delta t_{j-1}}{2}\hat{t}_{j-1}+\frac{\Delta t_{j}}{2}\hat{t}_{j}\right]\cdot\int_{t_{i-1/2}}^{t_{i+1/2}}\hat{t}'\frac{e^{-jkR_j}}{4\pi R_j}dt'+\frac{j\eta_o}{k_o}\left[\begin{array}{l}\dfrac{1}{\Delta t_{i-1}}\displaystyle\int_{t_{i-1}}^{t_i}\frac{e^{-jkR_j^c}}{4\pi R_j}dt'-\dfrac{1}{\Delta t_i}\displaystyle\int_{t_i}^{t_{i+1}}\frac{e^{-jkR_j^c}}{4\pi R_j}dt'\\ -\dfrac{1}{\Delta t_{i-1}}\displaystyle\int_{t_{i-1}}^{t_i}\frac{e^{-jkR_{j-1}^c}}{4\pi R_{j-1}}dt'+\dfrac{1}{\Delta t_i}\displaystyle\int_{t_i}^{t_{i+1}}\frac{e^{-jkR_{j-1}^c}}{4\pi R_{j-1}}dt'\end{array}\right]\right\}$$

- This can be expressed as a linear system of equations as:

$$\overline{\overline{Z}}\,\overline{\alpha}=\overline{f}_e$$

- This is used to solve for the unknown constant coefficient vector $\overline{\alpha}=\overline{\overline{Z}}^{-1}\overline{f}_e$

Evalauating the Integrals

- Computing the impedance matrix involves integrals of the form:
  - $\int_{t_i}^{t_{i+1}} \frac{e^{-jkR_j^c}}{4\pi R_j} dt'$
- Consider first, the self-term ($i = j$):
  - $\int_{t_j}^{t_{j+1}} \frac{e^{-jkR_j^c}}{4\pi R_j} dt' = 2\int_0^{\frac{\Delta t}{2}} \frac{e^{-jk\sqrt{t'^2+a^2}}}{4\pi\sqrt{t'^2+a^2}} dt'$
- It is expected that $\Delta t << \lambda$, therefore, $k\Delta t$ is small.
- One can then approximate:
  - $e^{-jk\sqrt{t'^2+a^2}} \approx 1 - jk\sqrt{t'^2+a^2}$
- Therefore:

$$\int_{t_j}^{t_{j+1}} \frac{e^{-jkR_j^c}}{4\pi R_j} dt' \approx 2\int_0^{\frac{\Delta t_j}{2}} \frac{1}{4\pi\sqrt{t'^2+a^2}} dt' - 2\int_0^{\frac{\Delta t_j}{2}} \frac{jk}{4\pi} dt' = \frac{1}{2\pi}\ln\left(\sqrt{1+\left(\frac{\Delta t_j}{2a}\right)^2} + \frac{\Delta t_j}{2a}\right) - \frac{jk\Delta t_j}{4\pi}$$

- Similarly,

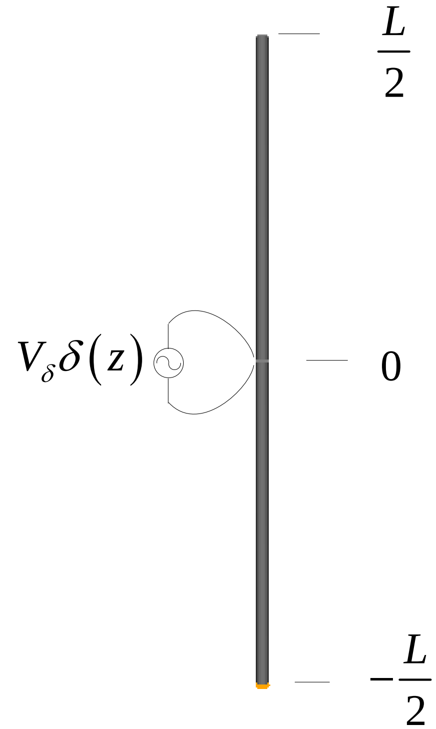
$$\int_{t_{j-1/2}}^{t_{j+1/2}} \hat{t}' \frac{e^{-jkR_j^c}}{4\pi R_j} dt' \approx \frac{\hat{t}'_{j-1}}{4\pi}\left(\ln\left(\sqrt{1+\left(\frac{\Delta t_{j-1}}{2a}\right)^2} + \frac{\Delta t_{j-1}}{2a}\right) - \frac{jk\Delta t_{j-1}}{2}\right) + \frac{\hat{t}_j}{4\pi}\left(\ln\left(\sqrt{1+\left(\frac{\Delta t_j}{2a}\right)^2} + \frac{\Delta t_j}{2a}\right) - \frac{jk\Delta t_j}{2}\right)$$

- For the non-self-terms, the integrals can be evaluated using a Gauss-Legendre quadrature rule

$$\int_{t_j}^{t_{j+1}} \frac{e^{-jkR_i^c}}{4\pi R_i} dt' \approx \frac{1}{4\pi} \sum_{q=1}^{N_q} \frac{e^{-jk\left|\vec{r}_i^{\,c} + \hat{\rho}_i a - \vec{r}(t_q)\right|}}{4\pi\left|\vec{r}_i^{\,c} + \hat{\rho}_i a - \vec{r}_j(t_q)\right|} \omega_q \Delta t_j$$

  - where t_q is the quadrature abscissa on the *j*-th source cell, $\vec{r}_j(t_q)$ is the position coordinate on the *j*-th source cell and ω_q is the quadrature weight.
- Typically a 4-point rule is sufficient to accurately evaluate the neighboring term.

Example: Thin-Wire Dipole Antenna



- Consider a thin-wire dipole antenna of length L
  - Center fed by a delta-gap source
- The thin wire is sub-divided into N linear segments
  - N is an even number (***N/2*** segs above and below source)
- Discrete EFIE:

$$V_\delta \delta_{\frac{N}{2},j} = \sum_{i=1}^{N} \alpha_i \left\{ jk_o \eta_o \Delta z \int_{z_{i-1/2}}^{z_{i+1/2}} \frac{e^{-jkR_j}}{4\pi R_j} dz' + \frac{j\eta_o}{\Delta z k_o} \left[\begin{array}{l} \int_{z_{i-1}}^{z_i} \frac{e^{-jkR_j^c}}{4\pi R_j^c} dz' - \int_{z_i}^{z_{i+1}} \frac{e^{-jkR_j^c}}{4\pi R_j^c} dz' \\ - \int_{z_{i-1}}^{z_i} \frac{e^{-jkR_{j-1}^c}}{4\pi R_{j-1}^c} dz' + \int_{z_i}^{z_{i+1}} \frac{e^{-jkR_{j-1}^c}}{4\pi R_{j-1}^c} dz' \end{array} \right] \right\}$$

  - where $R_j = \sqrt{(z_j - z')^2 + a^2}$ and $R_j^c = \sqrt{(z_j^c - z')^2 + a^2}$
  - For a 1-based indexing system ($j = 1..N$):
    - $z_j = -\frac{L}{2} + \Delta z \cdot (j-1), \quad z_j^c = -\frac{L}{2} + \Delta z \cdot \left(j - \frac{1}{2} \right)$
- Wire ends:
  - $I(-L/2) = 0,\ I(L/2) = 0$, therefore, $\alpha_1 = 0$ and $\alpha_N = 0$

MathCad Code

Reference the gauss legendre quadrature rules:

Reference:G:\Gedney\courses\ee525\Notes\GL_QuadRules.xmcd

$L := 0.47 \qquad k := 2\pi \qquad N := 40 \qquad dl := \frac{L}{N} \qquad a := 0.005 \qquad \frac{a}{L} = 0.010638298$

$z(n) := \frac{-L}{2} + n \cdot dl \qquad zmc(m) := \frac{z(m) + z(m+1)}{2} \qquad \eta := 376.730313461770655 4679$

$$apzSelf(b) := \left[\frac{1}{2\pi} \cdot \ln\left[\sqrt{1 + \left(\frac{dl}{2 \cdot a}\right)^2} + \frac{dl}{2a}\right] - \frac{j \cdot k \cdot dl}{4\pi}\right] \qquad exzSelf(b) := \frac{1}{2\pi} \cdot \int_0^{\frac{dl}{2}} \frac{e^{-j \cdot k \cdot \sqrt{z^2 + b^2}}}{\sqrt{z^2 + b^2}} \, dz$$

$apzSelf(a) = 0.159134191 - 5.875i \times 10^{-3} \qquad exzSelf(a) = 0.159023746 - 5.873588966i \times 10^{-3}$

$Zself := apzSelf(a)$

```
zmnP(m,n,Nq) :=  | int ← Zself  if  m = n
                 | otherwise
                 |   | int ← 0
                 |   | zq ← xq(Nq)
                 |   | wg ← wq(Nq)
                 |   | zm ← zmc(m)
                 |   | for i ∈ 0..Nq − 1
                 |   |   | zp ← z(n) + zq_i·dl
                 |   |   | Rm ← √((zm − zp)^2 + a^2)
                 |   |   | int ← int + (e^(−j·k·Rm) / Rm)·wg_i
                 |   | int ← int·(dl / 4π)
                 | int

zmnA(m,n,Nq) :=  | int ← Zself  if  m = n
                 | otherwise
                 |   | int ← 0
                 |   | zq ← xq(Nq)
                 |   | wg ← wq(Nq)
                 |   | zm ← z(m)
                 |   | for i ∈ 0..Nq − 1
                 |   |   | zp ← zmc(n − 1) + zq_i·dl
                 |   |   | Rm ← √((zm − zp)^2 + a^2)
                 |   |   | int ← int + (e^(−j·k·Rm) / Rm)·wg_i
                 |   | int ← int·(dl / 4π)
                 | int
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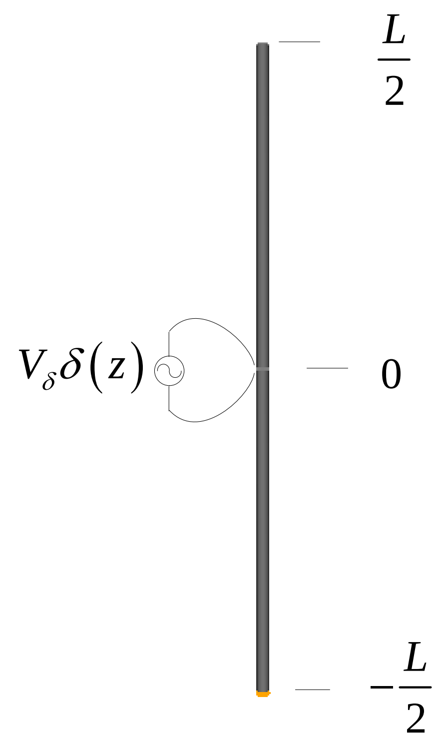

$Z := \left| \begin{array}{l} \text{for } i \in 1..N-1 \\ \quad \text{for } j \in 1..N-1 \\ \quad \left| \begin{array}{l} tmp_{i-1,j-1} \leftarrow \frac{\eta}{j \cdot k \cdot dl} \cdot [(zmnP(i-1, j-1, nq) - zmnP(i, j-1, nq)) - zmnP(i-1, j, nq) + zmnP(i, j, nq)] \\ tmp_{i-1,j-1} \leftarrow tmp_{i-1,j-1} + j \cdot k \cdot \eta \cdot dl \cdot (zmnA(i, j, nq)) \end{array} \right. \\ tmp \end{array} \right.$

$Nc := \frac{N}{2}$

Center fed delta-gap source:

$b := \left| \begin{array}{l} \text{for } i \in 1..N-1 \\ \quad tmp_{i-1} \leftarrow 0 \\ tmp_{Nc-1} \leftarrow 1 \\ tmp \end{array} \right.$

$Iz := Z^{-1} \cdot b$

Z =

	0	1	2	3
0	0.094561-832.093925i	0.116064+226.776704i	0.108685+126.447392i	0.108389+34.161997i
1	0.116064+226.776704i	0.094561-832.093925i	0.116064+226.776704i	0.108685+126.447392i
2	0.108685+126.447392i	0.116064+226.776704i	0.094561-832.093925i	0.116064+226.776704i
3	0.108389+34.161997i	0.108685+126.447392i	0.116064+226.776704i	0.094561-832.093925i
4	0.107975+14.012105i	0.108389+34.161997i	0.108685+126.447392i	0.116064+226.776704i
5	0.107445+7.177286i	0.107975+14.012105i	0.108389+34.161997i	0.108685+126.447392i
6	0.1068+4.211336i	0.107445+7.177286i	0.107975+14.012105i	...

Zero out the end nodes:

$\alpha := \left| \begin{array}{l} tmp_0 \leftarrow 0 \\ \text{for } i \in 1..N-1 \\ \quad tmp_i \leftarrow Iz_{i-1} \\ tmp_N \leftarrow 0 \\ tmp \end{array} \right.$

$Zin := \frac{1}{\alpha_{Nc}}$

$Zin = 76.297407357 + 4.8249523i$

$Re(Zin) = 76.297407357$

$i := 0..N$