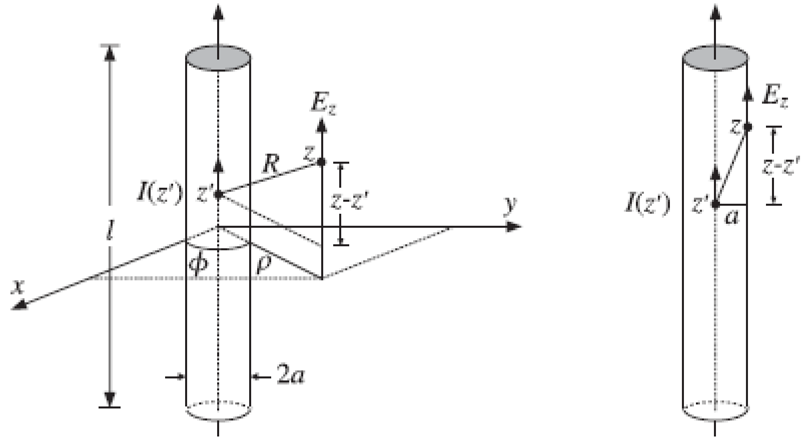
Thin Wire Antenna



- Consider a straight thin-wire antenna excited at the center by an arbitrary source
- Objective:
  - o Compute the fields scattered (or radiated) by the antenna
- Solution
  - o Use Hallen's equation as the base formulation
  - o Derive solution to Hallen's equation
  - o Use a method of moment discretization of Hallen's equation to approximate the thin-wire current

Deriving Hallen's Equation

- Hallen's equation for the straight thin wire is derived from a 1D form of the EFIE:
  - $\left[\frac{\partial^2}{\partial z^2} + k^2\right] A_z(z) = -\frac{jk}{\eta} E_z^{inc}(z)$
  - which is essentially representative of a thin-wire kernel
- Hallen posed that the solution to this equation can be derived using the Green's function method. To this end:
  - Solve the homogeneous equation
  - Solve the impulse response
- Homogeneous equation solution:
  - $\left[\frac{\partial^2}{\partial z^2} + k^2\right] A_z^h(z) = 0$
  - Has solutions:
    - $A_z^h(z) = C_1 e^{+jkz} + C_2 e^{-jkz}$

- The forced solution is based on the impulse response:
  - $\left[\frac{\partial^2}{\partial z^2}+k^2\right]G(z)=\delta(z)$
  - One form of solution to this equation can be expressed as:
    - $G(z)=C_3\sin\left(k|z|\right)$
  - Note that this is continuous across $z = 0$, but has discontinuous derivatives.
  - We need to solve for C_3. To do this, plug this back into the differential equation and integrate both sides:
    - $\lim_{\in\to 0}\int_{-\in}^{+\in}\left[\frac{\partial^2}{\partial z^2}+k^2\right]C_3\sin\left(k|z|\right)dz=\lim_{\in\to o}\int_{-\in}^{+\in}\delta(z)\,dz$
  - Differentiating, integrating by parts, leads to:
    - $\lim_{\in\to 0}kC_3\left[-\cos\left(k(-z)\right)\Big|_{-\in}^{0}+\cos\left(kz\right)\Big|_{0}^{+\in}+k\int_{-\in}^{+\in}\sin\left(k|z|\right)dz\right]=1$
  - Taking the limit $\in\to 0$
    - $kC_3\left[2+k\cdot 0\right]=1$
  - Therefore,
    - $C_3=\frac{1}{2k}$, and $G(z)=\frac{1}{2k}\sin\left(k|z|\right)$

- Combining the homogeneous and forced solution (impulse response)
  - $A_z(z) = C_1 e^{+jkz} + C_2 e^{-jkz} - \dfrac{jk}{\eta} E_z^{inc}(z) * G(z)$
  - $A_z(z) = C_1 e^{+jkz} + C_2 e^{-jkz} - \dfrac{jk}{\eta} \displaystyle\int_{-\frac{L}{2}}^{\frac{L}{2}} E_z^{inc}(z') G(z,z') dz'$
- We recall that if the current carries a current *I(z)*, the magnetic vector potential is also expressed as
  - $A_z(z) = \displaystyle\int_{\frac{-L}{2}}^{\frac{L}{2}} I(z') \frac{e^{-jkR_{TW}}}{4\pi R_{TW}} dz'$
  - where
    - $R_{TW} = \sqrt{(z-z')^2 + a^2}$
- Therefore, Hallen's equation is expressed as:
  - $\displaystyle\int_{\frac{-L}{2}}^{\frac{L}{2}} I(z') \frac{e^{-jkR_{TW}}}{4\pi R_{TW}} dz' = C_1 e^{+jkz} + C_2 e^{-jkz} - \frac{jk}{\eta} \int_{-\frac{L}{2}}^{\frac{L}{2}} E_z^{inc}(z') G(z,z') dz'$

Hallen's Equation

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} I(z')\frac{e^{-jkR_{TW}}}{4\pi R_{TW}}dz' = C_1 e^{+jkz} + C_2 e^{-jkz} - \frac{jk}{\eta}\int_{-\frac{L}{2}}^{\frac{L}{2}} E_z^{inc}(z')G(z,z')dz'$$

- Advantages of Hallen's equation:
  - Removed the derivatives of the scalar potential
    - Smoother kernel
  - Fredholm integral equation (improved conditioning)
- Disadvantages
  - Restricted to straight wires
  - Restricted to the thin-wire kernel
  - Two additional degrees of freedom to solve for
- Solution:
  - Solution can be derived via a method of moment expansion of the current and a moment testing.
  - Due to the smoothness of the kernel, one can choose pulse basis and point matching as the lowest order approximation

Solving Hallen's Equation with PulseBasis/Point Matching

- Discretization
  - Discretize the straight wire into *N* linear edges, each with length $\Delta L = L / N$
  - Expand the current with pulse basis functions weighted by unknown constant coefficients α_n
    - $I(z) \approx \sum_{n=1}^{N} \alpha_n P(z; z_n, z_{n+1})$
  - Perform the inner product of Hallen's equations with delta-test fns:
    - $T_m(z) \approx \delta\left(\left|z - z_m^c\right|\right)$, where z_m^c is the center of the *m*-th cell

$$\sum_{n=1}^{N} \alpha_n \int_{z_n}^{z_{n+1}} \frac{e^{-jkR_m}}{4\pi R_m} dz' = C_1 e^{+jkz_m} + C_2 e^{-jkz_m} - \frac{j}{2\eta} \int_{-\frac{L}{2}}^{\frac{L}{2}} E_z^{inc}(z') \sin\left(k\left|z_m^c - z'\right|\right) dz'$$

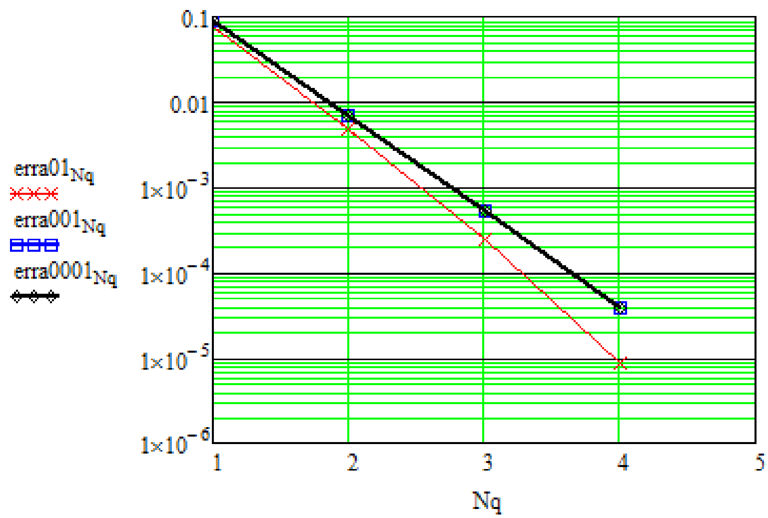
  - where
    - $R_m = \sqrt{\left(z_m^c - z'\right)^2 + a^2}$

- The impedance matrix can be constructed from:

- $$\int_{z_n}^{z_{n+1}} \frac{e^{-jkR_m}}{4\pi R_m} dz' = \begin{cases} \displaystyle\int_{z_n}^{z_{n+1}} \frac{e^{-jk\sqrt{\left(z_m^c - z'\right)^2 + a^2}}}{4\pi\sqrt{\left(z_m^c - z'\right)^2 + a^2}} dz', & m \neq n \\ \displaystyle\frac{1}{4\pi} \ln\left[\frac{\sqrt{1 + 4a^2 / \Delta L^2} + 1}{\sqrt{1 + 4a^2 / \Delta L^2} - 1}\right] - \frac{jk\Delta L}{4\pi}, & m = n \end{cases}$$

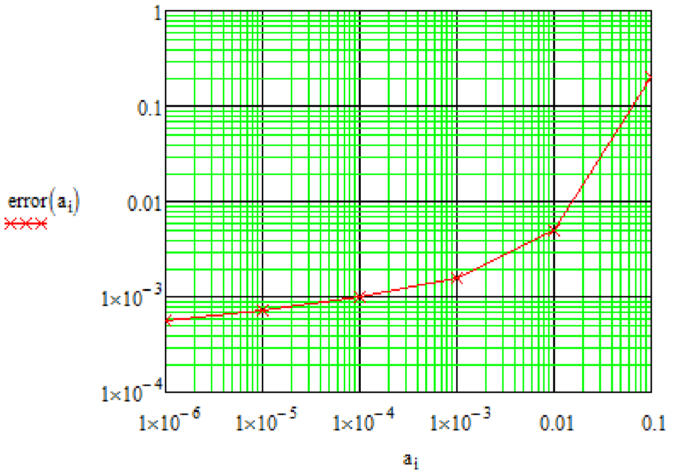
  - where the non-self-term can be computed via a Gauss-quadrature rule

- Error in the numerical quadrature evaluation of the non-self-terms:
  - $L = \lambda/2$, $N = 20$, and $a = 0.01L$, $0.001L$, $0.0001L$



- The self term is approximated as:
  - $$\int_{z_n}^{z_{n+1}} \frac{e^{-jkR_n}}{4\pi R_n} dz' \approx \frac{1}{4\pi} \ln\left[\frac{\sqrt{1+4a^2/\Delta L^2}+1}{\sqrt{1+4a^2/\Delta L^2}-1}\right] - \frac{jk\Delta L}{4\pi}$$

- Example:
  - $L = \lambda/2$, N = 10 segments ($\Delta L = 0.05\lambda$)
    - Error in the self-term approximation:



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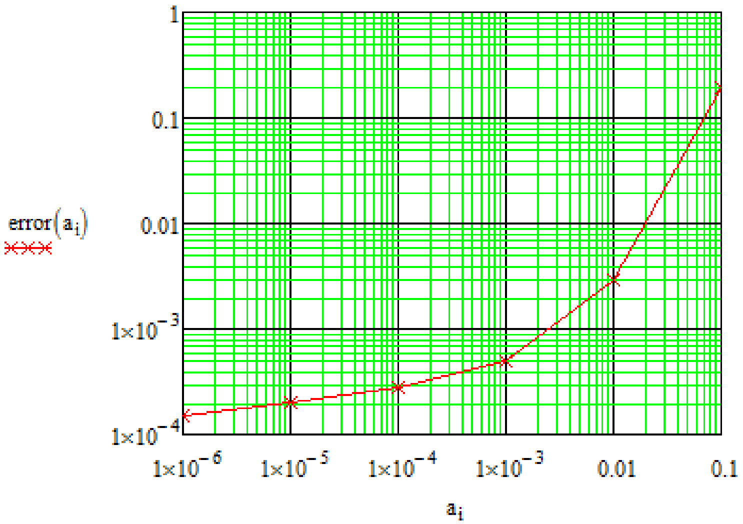

- Note: the impedance matrix leads to an $N \times N$ linear system of equations
  - However we have $N+2$ degrees of freedom!!
  - We need 2 more constraints
- Constrain that $I\left(-\frac{L}{2}\right)=0$, and $I\left(+\frac{L}{2}\right)=0$
  - In our discrete approximation:
    - $I\left(-\frac{L}{2}\right) \approx \sum_{n=1}^{N} \alpha_n P\left(-\frac{L}{2}; z_n, z_{n+1}\right)=0$, or $\alpha_1 = 0$
    - Similarly, $\alpha_N = 0$

- Next, define the contribution from the incident field
  - $b_m = -\frac{j}{2\eta} \int_{-\frac{L}{2}}^{\frac{L}{2}} E_z^{inc}(z') \sin\left(k\left|z_m^c - z'\right|\right) dz'$
    - The kernel is a smooth function, and thus can be evaluated with Gauss-quadrature, or analytically.
  - **Example**: Delta Gap Source: $E_z^{inc}(z') = V_o \delta(z')$

$$b_m = -\frac{j}{2\eta} V_o \sin\left(k\left|z_m^c\right|\right)$$

Discrete Hallen's Equation with PulseBasis/Point Matching

$$\int_{\frac{-L}{2}}^{\frac{L}{2}} I(z')\frac{e^{-jkR_{TW}}}{4\pi R_{TW}}dz' - C_1 e^{+jkz} - C_2 e^{-jkz} = -\frac{jk}{\eta}\int_{-\frac{L}{2}}^{\frac{L}{2}} E_z^{inc}(z')G(z,z')dz'$$

- Finally, Hallen's equation can be expressed in matrix form:

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ e^{+jkz_1^c} & Z_{1,1} & Z_{1,2} & \cdots & Z_{1,N} & e^{-jkz_1^c} \\ e^{+jkz_2^c} & Z_{2,1} & Z_{2,2} & \cdots & Z_{2,N} & e^{-jkz_2^c} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ e^{+jkz_N^c} & Z_{N,1} & Z_{N,2} & \cdots & Z_{N,N} & e^{-jkz_N^c} \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} -C_1 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \\ -C_2 \end{bmatrix} = -\frac{j}{2\eta}V_o \begin{bmatrix} 0 \\ \sin\left(k\left|z_1^c\right|\right) \\ \sin\left(k\left|z_2^c\right|\right) \\ \vdots \\ \sin\left(k\left|z_N^c\right|\right) \\ 0 \end{bmatrix}$$

  - Once we compute the solution vector, we can approximate the current as
    - $I(z) \approx \sum_{n=1}^{N} \alpha_n P(z; z_n, z_{n+1})$

Schur's compliment of the matrix equation

- The system matrix has zero diagonal elements, which can lead to ill-conditioning. This can be avoided by working with the Schur's compliment of the system matrix.
- Write the matrix equation as:
  - $\bar{\bar{Z}}\bar{\alpha} = \bar{\bar{s}}\bar{C} + \bar{b}$
    - where $\bar{\bar{s}}$ is a $N \times 2$ matrix expressed as $$\bar{\bar{s}} = \begin{bmatrix} e^{+jkz_1^c} & e^{-jkz_1^c} \\ e^{+jkz_2^c} & e^{-jkz_2^c} \\ \vdots & \vdots \\ e^{+jkz_N^c} & e^{-jkz_N^c} \end{bmatrix}$$
  - We can then solve:
    - $\bar{\alpha} = \bar{\bar{Z}}^{-1}\bar{\bar{s}}\bar{C} + \bar{\bar{Z}}^{-1}\bar{b}$
  - Then, express the constraint that $\alpha_1 = 0$ and $\alpha_N = 0$ in operator form:
    - $\bar{\bar{U}}\bar{\alpha} = 0$, where $\bar{\bar{U}} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}$
  - Therefore:
    - $\bar{\bar{U}}\bar{\alpha} = \bar{\bar{U}}\bar{\bar{Z}}^{-1}\bar{\bar{s}}\bar{C} + \bar{\bar{U}}\bar{\bar{Z}}^{-1}\bar{b} = 0$
  - Which can be used to solve for $\bar{\bar{C}}$
    - $\bar{C} = -\left[\bar{\bar{U}}\bar{\bar{Z}}^{-1}\bar{\bar{s}}\right]^{-1}\bar{\bar{U}}\bar{\bar{Z}}^{-1}\bar{b}$
      - Note that $[\bar{\bar{U}}\bar{\bar{Z}}^{-1}\bar{\bar{s}}]$ is a 2×2 matrix, and $\bar{\bar{U}}\bar{\bar{Z}}^{-1}\bar{b}$ is a 2×1 vector

- Finally:
  - $\overline{\alpha} = \overline{\overline{Z}}^{-1}\left[\overline{\overline{I}} - \overline{\overline{s}}\left[\overline{\overline{U}}\overline{\overline{Z}}^{-1}\overline{\overline{s}}\right]^{-1}\overline{\overline{U}}\overline{\overline{Z}}^{-1}\right]\overline{b}$
- Example:
  - $L = 0.47\ \lambda,\ a = 0.005\lambda,\ N = 81$

$$Nc = 40 \qquad \alpha_{Nc} = 0.013402396 + 4.939504807i \times 10^{-4}$$

$$Zin := \frac{1}{\alpha_{Nc}} = 74.512310591 - 2.746179902i$$