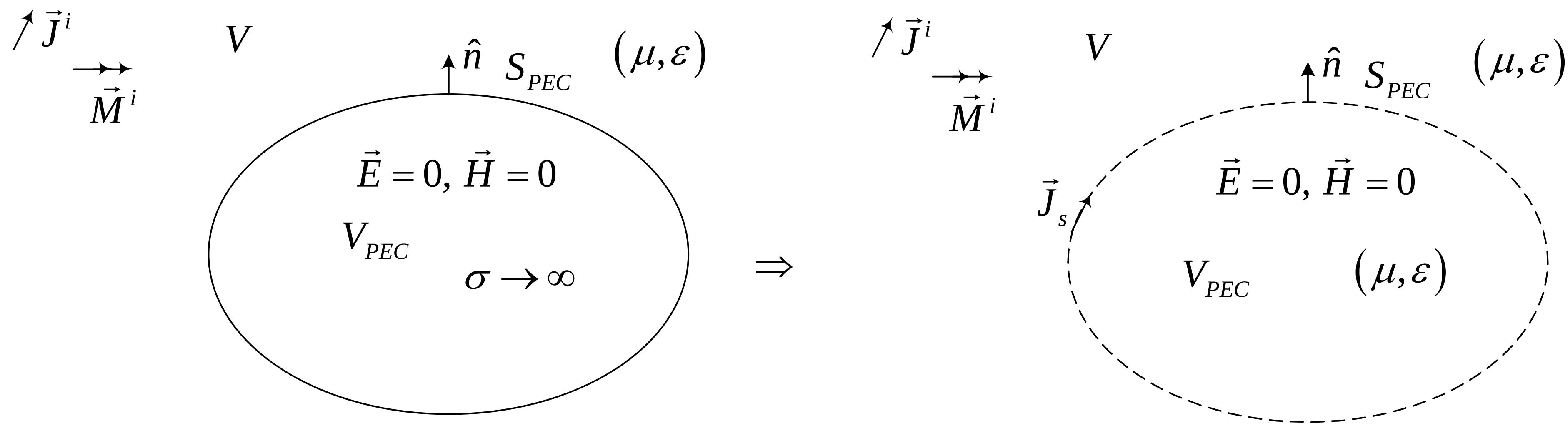
Equivalence

We derived using Green's Second Identity, an "Equivalence Principal" for the *scattering* of an electromagnetic field by a PEC object.



It was determined that the superposition of the equivalent surface currents and the original impressed current sources exactly reproduce the fields of the impressed currents radiating in the presence of the PEC object.

The equivalent current density is: $\vec{J}_s = \hat{n}\times\vec{H}^t\Big|_{S_{PEC}}$

The problem is that $\vec{H}^t$ is not known on S_{PEC}! Thus, we still need to devise a way to solve for either $\vec{H}^t$ or $\vec{J}_s$.

Setting up the Boundary Value Problem

- Since Green's second identity (GSI) can be used to exactly predict the fields due to the sources radiating in the presence of the PEC object, we can then use it to *solve* for the tangential fields on the boundary of the object.

- How?
  - By enforcing the boundary conditions on the surface of the object.

- Recall the boundary conditions governing the tangential fields on the surface of a PEC:

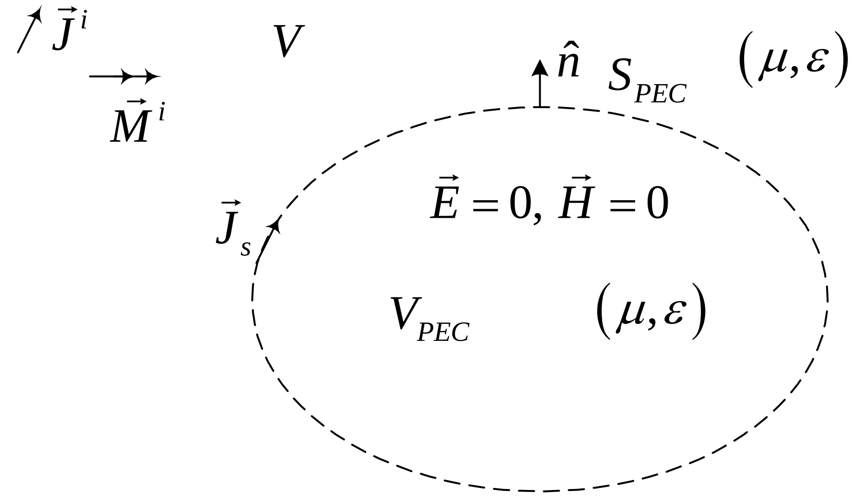
$$\hat{n} \times \vec{E}^{t}\Big|_{S_{PEC}} = 0$$

$$\hat{n} \times \vec{H}^{t}\Big|_{S_{PEC}} = \vec{J}_s$$

- These substitutions have already been used to modify GSI to represent the equivalent current problem.

- However, since $\vec{J}_s$ is still unknown, it suffices that the solution for $\vec{J}_s$ must explicitly lead to these boundary conditions.

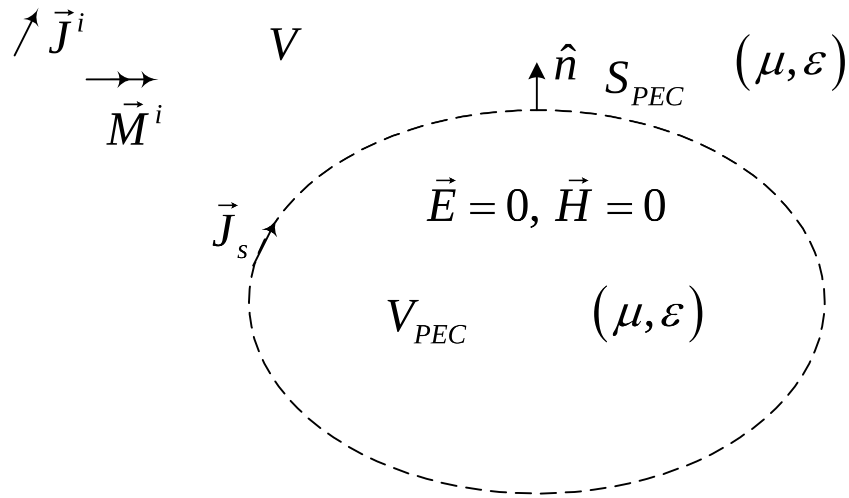
The Electric Field Integral Equation (EFIE)



- The Electric Field Integral Equation (better known as the "EFIE") is derived by enforcing the zero tangential electric field boundary condition:
  - $\hat{n} \times \vec{E}^t \Big|_{S_{PEC}} = 0$
- Note that $\vec{E}^t$ is radiated by both the impressed current densities ($\vec{E}^i$) and the equivalent current density ($\vec{E}^s$).
- Therefore, we can write:
  - $\hat{n} \times \left(\vec{E}^i + \vec{E}^s \right) \Big|_{S_{PEC}} = 0$, or
  - $-\hat{n} \times \vec{E}^i \Big|_{S_{PEC}} = \hat{n} \times \vec{E}^s \Big|_{S_{PEC}}$

- This is the EFIE!

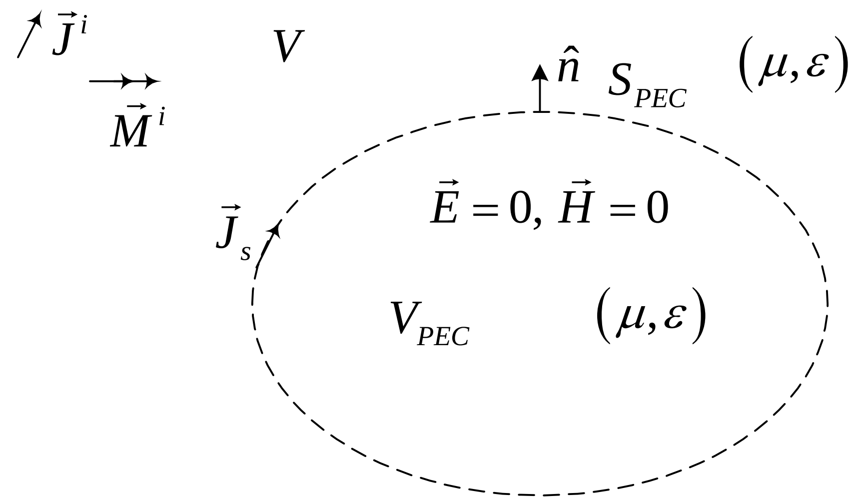
The EFIE



- The incident field $\vec{E}^i$ is simply the fields radiated by the impressed currents in a homogeneous free space.
- The scattered field $\vec{E}^s$ is the field radiated by $\vec{J}_s$, which is still unknown.
- Using vector potentials, we can more immediately arrive at GSI:
  - $\vec{E}^s(\mathbf{r}) = -j\omega\mu\vec{A}(\mathbf{r}) + \frac{1}{j\omega\varepsilon}\nabla\nabla\cdot\vec{A}(\mathbf{r})$, or
  - $\vec{E}^s(\mathbf{r}) = -j\omega\mu\iint_{S_{PEC}}\vec{J}_s(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds' + \frac{1}{j\omega\varepsilon}\nabla\nabla\cdot\iint_{S_{PEC}}\vec{J}_s(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds'$
- The EFIE, then more explicitly becomes:
  - $-\hat{n}\times\vec{E}^i(\mathbf{r}) = -\hat{n}\times j\omega\mu\iint_{S_{PEC}}\vec{J}_s(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds' + \hat{n}\times\frac{1}{j\omega\varepsilon}\nabla\nabla\cdot\iint_{S_{PEC}}\vec{J}_s(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds'$
    - where, $\mathbf{r}\in S_{PEC}$

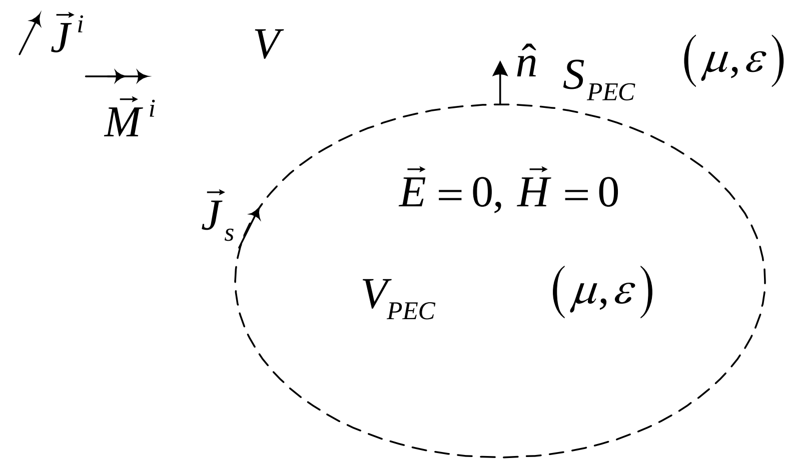
- The EFIE can then be used to solve for $\vec{J}_s$.

The Magnetic Field Integral Equation (MFIE)



- The Magnetic Field Integral Equation (better known as the "MFIE") is derived by enforcing the tangential boundary condition of the magnetic field:
  - $\hat{n}\times\left(\vec{H}^t - 0\right)\Big|_{S^+_{PEC}} = \vec{J}_s$
- Note that $\vec{H}^t$ can also be expressed as a superposition of incident and scattered fields
- Therefore, we can write:
  - $\hat{n}\times\left(\vec{H}^i + \vec{H}^s\right)\Big|_{S^+_{PEC}} = \vec{J}_s$, or
  - $\hat{n}\times\vec{H}^i\Big|_{S^+_{PEC}} = \vec{J}_s - \hat{n}\times\vec{H}^s\Big|_{S^+_{PEC}}$
- This is the MFIE

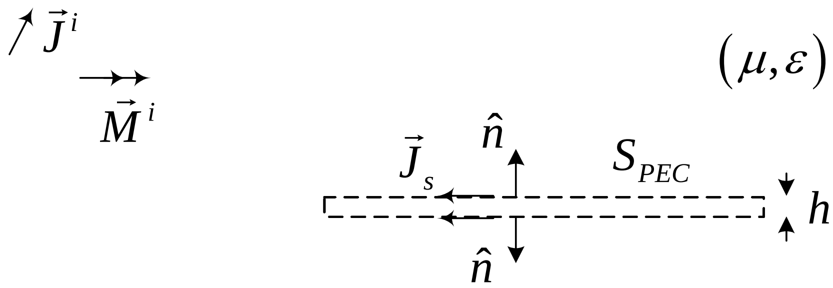
The MFIE



- The incident field $\vec{H}^i$ is the magnetic field radiated by the impressed currents in a homogeneous free space.
- The scattered field $\vec{H}^s$ is the field radiated by $\vec{J}_s$, which is still unknown.
- Using vector potentials,
  - $\vec{H}^s(\mathbf{r}) = \nabla\times\vec{A}(\mathbf{r})$, or
  - $\vec{H}^s(\mathbf{r}) = \nabla\times\iint_{S_{PEC}}\vec{J}_s(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds'$
- The MFIE, then more explicitly becomes:
  - $\hat{n}\times\vec{H}^i(\mathbf{r}) = \vec{J}_s(\mathbf{r}) - \hat{n}\times\nabla\times\iint_{S_{PEC}}\vec{J}_s(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds'$
    - where, $\mathbf{r}\in S_{PEC}^{+}$ (just inside *V*)

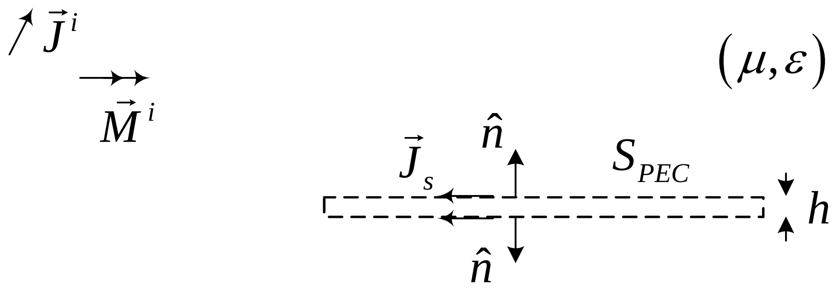
- The MFIE can also be used to solve for $\vec{J}_s$.

The EFIE for Thin PEC Objects



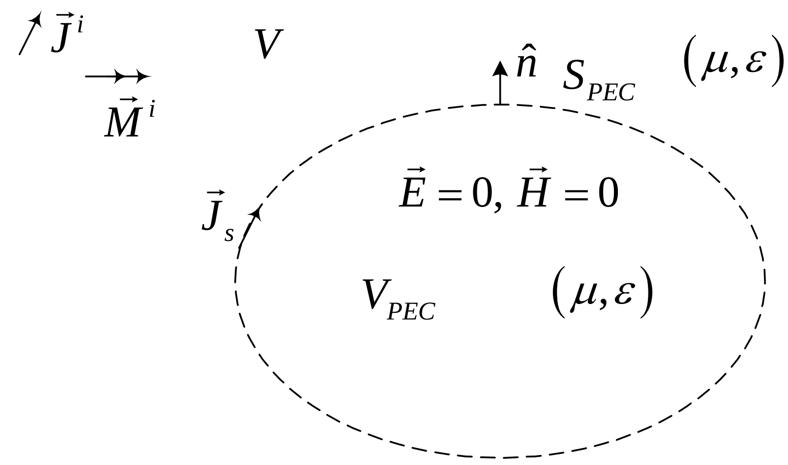
- Consider a very thin PEC object situated in an unbounded medium.
- The EFIE is written as:
  - $-\hat{n}\times\vec{E}^i(\mathbf{r}) = \hat{n}\times\vec{E}^s(\mathbf{r})$, where, $\mathbf{r}\in S_{PEC}$
- In the $\lim h\rightarrow 0$, the upper and lower surfaces effectively add, and the end surface has no contribution, leading to:
  - $$-\hat{n}\times\vec{E}^i(\mathbf{r}) = -j\omega\mu\hat{n}\times\iint_{S_{open}}\left(\vec{J}_s^{+}+\vec{J}_s^{-}(\mathbf{r}')\right)G(\mathbf{r},\mathbf{r}')ds' + \frac{\hat{n}\times}{j\omega\varepsilon}\nabla\nabla\cdot\iint_{S_{open}}\left(\vec{J}_s^{+}+\vec{J}_s^{-}(\mathbf{r}')\right)G(\mathbf{r},\mathbf{r}')ds'$$
    - where S_{open} is the open PEC surface representing the top & bottom surfaces
    - Note, this is justifiable b/c $\vec{E}^t$ & $\vec{E}^i$ are continuous, thus so is $\vec{E}^s$
- Which leads to:
  - $$-\hat{n}\times\vec{E}^i(\mathbf{r}) = -j\omega\mu\hat{n}\times\iint_{S_{open}}\underset{\sim}{\vec{J}}_s(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds' + \frac{\hat{n}\times}{j\omega\varepsilon}\nabla\nabla\cdot\iint_{S_{open}}\underset{\sim}{\vec{J}}_s(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds'$$
  - where $\underset{\sim}{\vec{J}}_s(\mathbf{r}') = \vec{J}_s^{+}+\vec{J}_s^{-}(\mathbf{r}')$, which is the superposition of the top & bottom currents

The MFIE for Thin PEC Objects



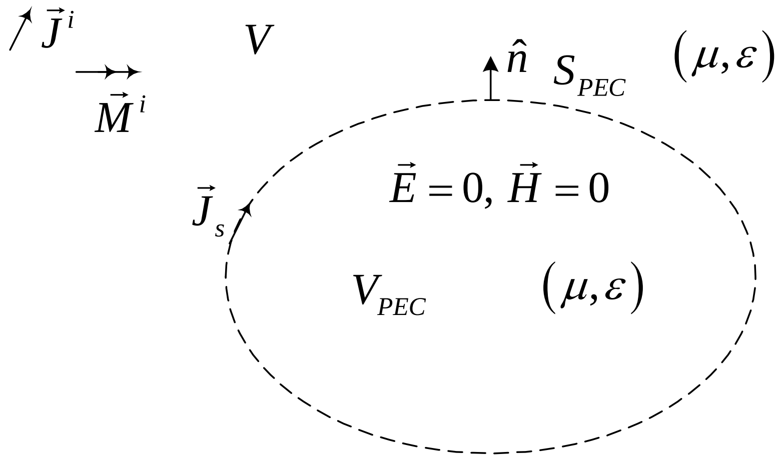
- This exercise is repeated for the MFIE:
  - $\hat{n}\times\vec{H}^i(\mathbf{r}^+)=\hat{n}\times\vec{J}_s(\mathbf{r}^+)-\hat{n}\times\nabla\times\iint_{S_{PEC}}\vec{J}_s(\mathbf{r}')G(\mathbf{r}^+,\mathbf{r}')ds'$
- Problem:
  - The magnetic field $\vec{H}^s$ undergoes a step discontinuity between the upper and lower surfaces. Thus, we cannot simply combine the upper and lower surface integrals.
- Conclusion:
  - The MFIE is not valid for infinitesimally thin objects
  - Furthermore, b/c of the step discontinuity in $\vec{H}^s$, the MFIE becomes ill-conditioned for very thin objects
    - Why: $\vec{J}_s(\mathbf{r}^+)$ is clearly different on the top and bottom. However, the MFIE equation becomes linearly dependent for $\mathbf{r}^+$ on the top and bottom surface in the $\lim h\to 0$
- Therefore, the MFIE is only valid for *closed* surfaces. And can be ill-conditioned for very long and thin conducting objects.

Interior Resonance Issues



- Consider a large, closed, PEC object.
- Either the EFIE or the MFIE can be used to solve for the surface current densities.
- Strict uniqueness requires the medium to be lossy.
- A PEC is actually lossless.
- There exist discrete frequencies at which the EFIE or the MFIE actually break down.
- These frequencies correspond to natural *interior* resonances of the interior problem.
  - o These are the resonant frequencies of the interior volume if it is treated as a cavity bound by the PEC surface S_{PEC}.
- These cavity modes satisfy exactly the same boundary conditions
  - o $\hat{n}\times\left(\vec{H}^{t-}-0\right)\Big|_{S_{PEC}}=\vec{J}_s$, & $\hat{n}\times\left(\vec{E}^{t-}-0\right)\Big|_{S_{PEC}}=0$
- Cavity resonances occur only at discrete frequencies. At these frequencies, the interior fields are non-zero. However, they satisfy exactly the same boundary conditions as the exterior problem.
- They lead to a different solution for the surface current density, and therefore corrupt the solution.

The Combined Field Integral Equation



- We can avoid interior resonances if we apply both the EFIE and MFIE simultaneously.
  - This is referred to as the Combined Field Integral Equation (CFIE).
- The CFIE is a weighted average of the EFIE and MFIE:
  - CFIE = α EFIE + $(1-\alpha)\eta$ MFIE
  - where α is a constant, typically chosen to be $(0 \le \alpha \le 1)$
- The CFIE is more explicitly written as:

$$-\alpha\hat{n}\times\vec{E}^i(\mathbf{r})-(1-\alpha)\eta\hat{n}\times\vec{H}^i(\mathbf{r}^+)=-(1-\alpha)\eta\vec{J}_s(\mathbf{r}^+)-\hat{n}\times j\omega\mu\alpha\iint_{S_{PEC}}\vec{J}_s(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds'+$$

$$\hat{n}\times\frac{\alpha}{j\omega\varepsilon}\nabla\nabla\cdot\iint_{S_{PEC}}\vec{J}_s(\mathbf{r}')G(\mathbf{r},\mathbf{r}')ds'+(1-\alpha)\eta\,\hat{n}\times\nabla\times\iint_{S_{PEC}}\vec{J}_s(\mathbf{r}')G(\mathbf{r}^+,\mathbf{r}')ds'$$

  - where $\eta=\sqrt{\mu/\varepsilon}$ is the characteristic wave impedance of the host medium.
- The CFIE is one way to avoid interior resonances. Other means to do so, such as the Augmented Field Integral Equation, will be discussed later.