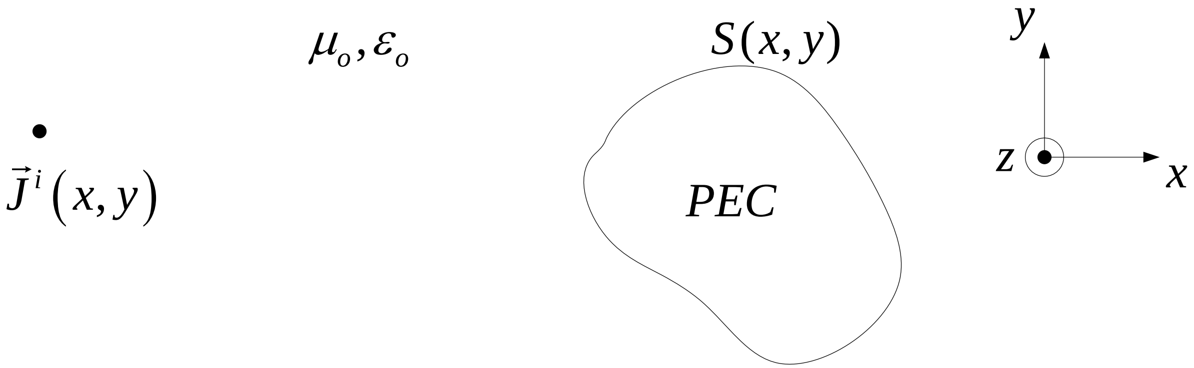
TM$_z$-Scattering by a PEC Cylinder – EFIE Solution



- Consider an electric current density $\vec{J}^i(x,y) = \hat{z}J_z^i$ that is infinitely long and invariant w.r.t. the *z*-axis radiating in the presence of an infinitely long PEC cylinder defined by the surface $S(x,y)$ which is situated in an infinite, homogeneous, unbounded media.
- Objective:
  - Compute the currents induced on the PEC cylinder
  - Compute the field scattered by the PEC Cylinder
- Solution
  - Pose the equivalent problem
  - Establish the Electric Field Integral Equation for this problem
  - Solve for the induced currents via the Method of Weighted Residuals
  - Compute the scattered electric field from the induced currents

Polarization

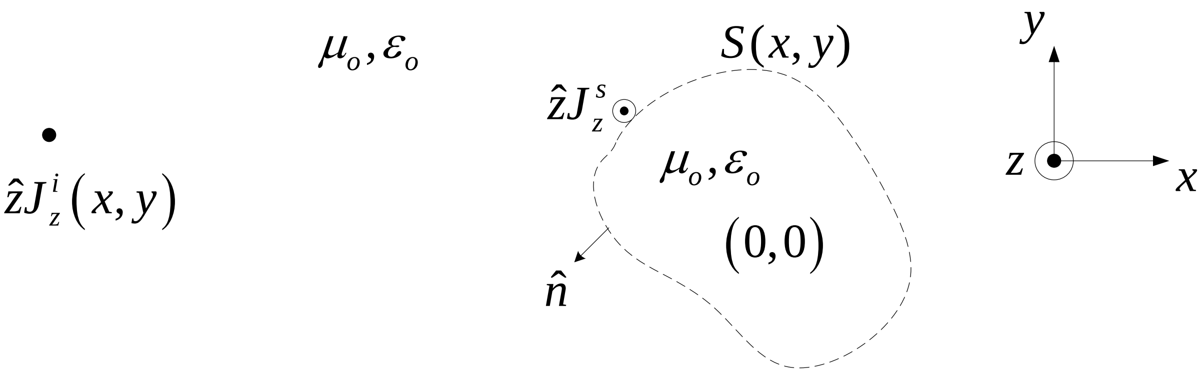
- Note that $\vec{J}^{\,i}$ is *z*-directed, and infinite and invariant along *z*.
  - o $\vec{J}^{\,i} = \hat{z} J_z^i(x,y)$
- From vector potential theory:
  - o $\vec{A} = \hat{z}\psi_m(x,y)$
- Therefore, the field radiated by the current density is TM$_z$ polarized.
  - o Thus:

$$\boxed{\begin{array}{ll} E_x = 0 & H_x = \dfrac{\partial}{\partial y}\psi_m \\ E_y = 0 & H_y = -\dfrac{\partial}{\partial x}\psi_m \\ E_z = -j\omega\mu\psi_m & H_z = 0 \end{array}}$$

- Since the cylinder is also infinite and invariant along the *z*-direction, the scattered field is TM$_z$ polarized.
- Also, since *H* is transverse only, and *z*-invariant, then only a *z*-directed current will be induced on the surface of the PEC cylinder.

The Equivalent Problem

$$\left(\vec{E}^{inc}+\vec{E}^{scat},\vec{H}^{inc}+\vec{H}^{scat}\right)$$



- Following Green's second identity, we can place an equivalent current density
$$\vec{J}_s=\hat{n}\times\vec{H}^{tot}\Big|_S=\hat{z}J_z^s$$
on the surface of the PEC that is effectively radiating in a homogeneous free space.
- The equivalent current radiates the scattered field in the region exterior to *S*, and the negative of the incident field in the region interior to *S*.
- The electric field integral equation (EFIE) was derived by enforcing the physical boundary condition of the tangential electric field on the surface *S* of the PEC:
$$\hat{n}\times\vec{E}^{tot}\Big|_S=0,$$
or,
$$\hat{n}\times\vec{E}^{inc}\Big|_S=-\hat{n}\times\vec{E}^{scat}\Big|_S$$

- Since $\vec{J}_s = \hat{z} J_z^s(x,y)$, the scattered field is TM_z-polarized. Hence,

$$\vec{E}^s = \hat{z} E_z^s = -j\omega\mu_o A_z$$

  - where,

$$A_z(\vec{r}) = \int_C G^{2D}(\vec{r},\vec{r}')\, J_z^s(\vec{r}')\, dl$$

  - where, *C* represents the two-dimensional contour defining the cross-section of *S*.
- Finally, the TM_z-polarized EFIE is expressed as:

$$\boxed{E_z^{inc}(\vec{r})\Big|_{\vec{r}\in C} = +j\omega\mu_o \int_C G^{2D}(\vec{r},\vec{r}')\, J_z^s(\vec{r}')\, dl}$$

which is defined in a z = constant plane.

- Defining the 2D Green's function, this can more exactly be expressed as:

$$\boxed{E_z^{inc}(\vec{r})\Big|_{\vec{r}\in C} = +j\omega\mu_o \int_C \frac{1}{4j} H_o^{(2)}\left(k_o\left|\vec{r}-\vec{r}'\right|\right) J_z^s(\vec{r}')\, dl}$$

The Method of Weighted Residuals Solution

$$\boxed{E_z^{inc}(\vec{r})\Big|_{\vec{r}\in C} = +j\omega\mu_o \int_C \frac{1}{4j} H_o^{(2)}\left(k_o\left|\vec{r}-\vec{r}'\right|\right) J_z^s(\vec{r}')\,dl}$$

- We can express this in the operator form:

$$f = \mathcal{L}x$$

  - $f = E_z^{inc}, \quad x = J_z^S, \& \quad \mathcal{L}x = j\omega\mu_o \int_C \frac{1}{4j} H_o^{(2)}\left(k_o\left|\vec{r}-\vec{r}'\right|\right) J_z^s(\vec{r}')\,dl$

- Recall, to solve this via the method of weighted residuals:
  - Expand the current density via a *basis function* space $\{\phi_i\}$ with support on C that spans a polynomial space of order p defined on C.
  - This is done here in parametric coordinates:

$$J_z^s(l) = \sum_{i=1}^{N} \alpha_i \phi_i(l)$$

    - The basis function space interpolates J over C
    - The quality of this interpolation is based on the choice of ϕ_i and N.
  - Expanding this within the operator:

$$E_z^{inc}(\vec{r})\Big|_{\vec{r}\in C} \approx +j\omega\mu_o \sum_{i=1}^{N} \alpha_i \int_C \frac{1}{4j} H_o^{(2)}\left(k_o\left|\vec{r}-\vec{r}'(l')\right|\right) \phi_i(l')\,dl$$

$$E_z^{inc}\left(\vec{r}\right)\Big|_{\vec{r}\in C} \approx +j\omega\mu_o\sum_{i=1}^{N}\alpha_i\int_C \frac{1}{4j}H_o^{(2)}\left(k_o\left|\vec{r}-\vec{r}'\left(l'\right)\right|\right)\phi_i\left(l'\right)dl$$

- At this point, the α_i are constant unknown coefficients (yet to be solved from this equation) and $\vec{r}$ is any arbitrary point on *C*.
- To formulate a linear system of equations, we will compute *N*-moments of the EFIE operator
  - Introduce a set *test function* space with support on *C* that spans polynomial space $\left\{\varphi_j\right\}$ of order *d* defined on *C*.
  - Compute the inner product of the EFIE with each of the ψ_i:

$$\left\langle\varphi_j, E_z^{inc}\right\rangle \approx +j\omega\mu_o\sum_{i=1}^{N}\alpha_i\left\langle\varphi_j,\int_C \frac{1}{4j}H_o^{(2)}\left(k_o\left|\vec{r}-\vec{r}'\left(l'\right)\right|\right)\phi_i\left(l'\right)dl\right\rangle$$

  - where,

$$\left\langle f, g\right\rangle = \int_C f\left(l\right)g\left(l\right)dl$$

- This can be expressed now as a linear system of equations:

$$\overline{f} = \overline{\overline{Z}}\,\overline{\alpha}$$

- where,

$$f_j = \left\langle \varphi_j, E_z^{inc} \right\rangle = \int_C \varphi_j(l) E_z^{inc}\left(\vec{r}(l)\right) dl$$

- is the forcing vector,

$$Z_{j,i} = \int_C \varphi_j(l) \int_C \frac{1}{4j} H_o^{(2)}\left(k_o \left|\vec{r}(l) - \vec{r}'(l')\right|\right) \phi_i(l')\, dl dl$$

- is referred to as the *impedance matrix*. Assuming there are *N* testing functions, this is an $N \times N$ dimensional matrix
  - Note that the matrix is fully populated

- and $\overline{\alpha}$ is the vector of unknown coefficients.

- The approximate solution for the surface currents is derived via:

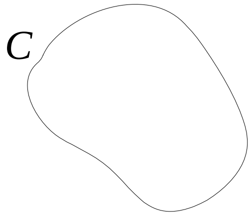
$$\overline{\alpha} = \overline{\overline{Z}}^{-1}\overline{f}$$

- Given this solution, the scattered field can be predicted at any point as:

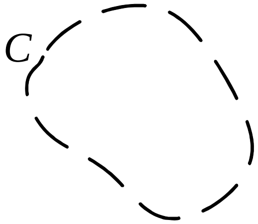
$$E_z^{scat}(\vec{r}) \approx -j\omega\mu_o \sum_{i=1}^{N} \alpha_i \int_C \frac{1}{4j} H_o^{(2)}\left(k_o \left|\vec{r} - \vec{r}'(l')\right|\right) \phi_i(l')\, dl$$

Entire-Domain and Sub-Domain Function Spaces

- The Method of Weighted Residual solution (referred to in CEM as the “Method of Moment” solution), requires the expansion of J via a known set of *basis functions* with support on C.
    - This requires knowledge of a description of C
        - This can be expressed in a parametric space
    - This then requires an expansion of the basis functions within this parametric space.
- We can define two classes of basis function spaces defined on C
    - *Entire Domain Basis Function Space*:
        - The basis functions span the entire support of C.
        - The basis are overlapping, and are complete to some polynomial order

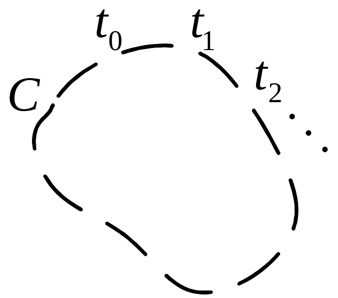


-
    - *Sub-Domain Basis Function Space:*
        - The global contour C is broken up into sub-intervals.
        - The union of all the sub-intervals defines C

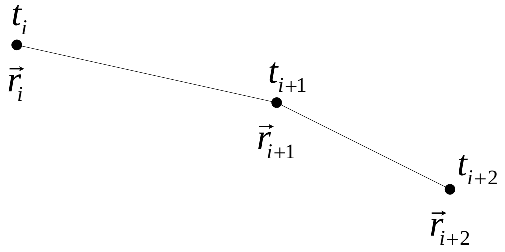


Sub-Sectional Approximation of *C*

- The contour *C* can be approximated via a series of line segments.



- Each line segment is bound by two vertices (t_i, t_{i+1}):
  - For example: assume a piece-wise linear approximation of *C*:



    - where, $\vec{r}_i = \vec{r}\left(t_i\right)$
  - We can express the position vector along the *i*-th segment as:

$$\vec{r}(t) = \vec{r}_i + \left(\vec{r}_{i+1} - \vec{r}_i\right)\frac{t - t_i}{t_{i+1} - t_i}$$

- Our basis functions can then be expressed in a closed form over each sub-interval as $\phi\left(t - t_i\right)$

Method of Moment Solution of the TMz EFIE via "Pulse-Basis" and "Point-Matching"

- Basis Functions:
  - Assume a piecewise linear approximation of the contour *C*.
  - Assume over each segment the current to be constant:

$$\phi_i(t) = \begin{cases} 1, & t_i \le t \le t_{i+1} \\ 0, & \text{else} \end{cases}$$

    - The current is thus approximate via "pulse" basis functions
      - These constitute a piecewise constant approximation of the current

- Testing Functions:
  - Choose "Point" test functions (i.e., delta-functions):

$$\varphi_j(t) = \delta\left(t - t_j^c\right)$$

    - where t_j^c is the center of the *j*-th segment:

$$t_j^c = \frac{t_j + t_{j+1}}{2}$$

Formulation: TMz EFIE via “Pulse-Basis” and “Point-Matching”

- Evaluate the Reaction from the MOM formulation:

$$\left\langle \delta\left(\left|\vec r-\vec r_j^{\,c}\right|\right), E_z^{inc}\right\rangle \approx +j\omega\mu_o\sum_{i=1}^{N}\alpha_i\left\langle \delta\left(\left|\vec r-\vec r_j^{\,c}\right|\right), \int_C \frac{1}{4j}H_o^{(2)}\left(k_o\left|\vec r-\vec r'(t')\right|\right)\phi_i(t')\,dt'\right\rangle$$

$$E_z^{inc}\left(\vec r_j^{\,c}\right) \approx +j\omega\mu_o\sum_{i=1}^{N}\alpha_i\int_C \frac{1}{4j}H_o^{(2)}\left(k_o\left|\vec r_j^{\,c}-\vec r'(t')\right|\right)\phi_i(t')\,dt'$$

  - For $j\in 1..N$. This leads to the linear system of equations:

$$\bar e^{inc} = \bar{\bar Z}\bar\alpha$$

  - Evaluating the *j*,*i*-th entry of the impedance Matrix

$$\boxed{Z_{j,i} = jk_o\eta_o\int_{t_i}^{t_{i+1}} \frac{1}{4j}H_o^{(2)}\left(k_o\left|\vec r_j^{\,c}-\vec r'(t')\right|\right)dt'}$$

    - where $\phi_i = 1$ over the *i-th* segment, and 0 everywhere else.

  - and

$$\boxed{e_j^{inc} = E_z^{inc}\left(\vec r_j^{\,c}\right)}$$

Computing the Impedance Matrix

$$\boxed{Z_{j,i} = \frac{k_o\eta_o}{4}\int_{t_i}^{t_{i+1}} H_o^{(2)}\left(k_o\left|\vec{r}_j^{\,c} - \vec{r}'(t')\right|\right)dt'}$$

- The impedance matrix entries will be evaluated via numerical integration
  - recall

$$\vec{r}(t') = \vec{r}_i + \left(\vec{r}_{i+1} - \vec{r}_i\right)\frac{t' - t_i}{t_{i+1} - t_i}, \quad t' \in t_i, t_{i+1}$$

- Initially, assume that $j \neq i$, such that the integrand is smooth
  - Can evaluate the integral using Gauss-Legendre Quadrature Integration
- To interpolate the currents to sufficient accuracy, the segments must be sufficiently small (i.e., $\Delta t << \lambda$).
  - We can approximate the integral with a 1-point quadrature rule (a mid-point rule):

$$Z_{j,i} \approx \frac{k_o\eta_o\Delta t_i}{4} H_o^{(2)}\left(k_o\left|\vec{r}_j^{\,c} - \vec{r}_i^{\,c}\right|\right)$$

    - where Δt = the width of the segment (i.e.,

$$\Delta t_i = \left|\vec{r}\left(t_{i+1}\right) - \vec{r}\left(t_i\right)\right| = \left|\vec{r}_{i+1} - \vec{r}_i\right|$$

- More accuracy can be realized via higher-order quadrature rules

Evaluating the "Self-Term" (i = j)

- When the field point lies on the source segment (i.e., $j = i$), the integrand is singular:

$$Z_{j,i} = \frac{k_o \eta_o}{4} \int_{t_i}^{t_{i+1}} H_o^{(2)}\left(k_o \left|\vec{r}_j^{\,c} - \vec{r}'(t')\right|\right) dt'$$

  - The Hankel function has a log singularity as $\vec{r}'(t') \to \vec{r}_i^{\,c}$.
  - The integral can be expressed as:

$$Z_{i,i} = \frac{k_o \eta_o}{4} \int_{t_i}^{t_i^c} H_o^{(2)}\left(k_o \left|\vec{r}_i^{\,c} - \vec{r}'(t')\right|\right) dl + \frac{k_o \eta_o}{4} \int_{t_i^c}^{t_{i+1}} H_o^{(2)}\left(k_o \left|\vec{r}_i^{\,c} - \vec{r}'(t')\right|\right) dt'$$

    - And we can apply a lin-log quadrature rule over each segment.
    - This can provide controlled accuracy

- We can also apply a small argument approximation for the singular part of the Hankel function
  - We can expand the Hankel function into its real and imaginary parts:

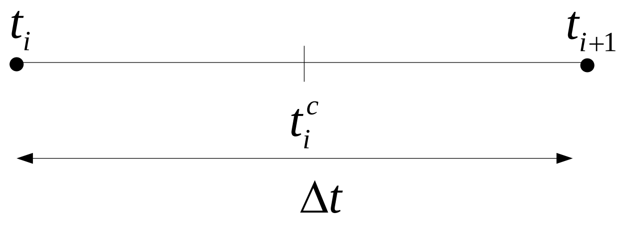
$$H_0^{(2)}\left(k_o R\right) = \underbrace{J_o\left(k_o R\right)}_{\text{regular}} - j\underbrace{Y_o\left(k_o R\right)}_{\text{singular}}$$

  - Applying a small argument approximation (c.f., Abromowitz & Stegun):

$$\lim_{k_o R \to 0} H_0^{(2)}\left(k_o R\right) \approx 1 - j\frac{2}{\pi}\ln\left(\frac{\gamma k_o R}{2}\right)$$

    - where, $\gamma = 1.781072418...$ is Euler's number.

    - Then, integrating over the line segment:



$$Z_{i,i} \approx 2\frac{k_o \eta_o}{4}\left\{\int_0^{\Delta t/2} 1 - j\frac{2}{\pi}\ln\left(\frac{\gamma k_o t'}{2}\right)dt'\right\}$$

- The integral can be derived in closed form as:

$$Z_{i,i} \approx 2\frac{k_o\eta_o}{4}\left\{\int_0^{\Delta t_i/2} 1 - j\frac{2}{\pi}\ln\left(\frac{\gamma k_o t'}{2}\right)dt'\right\}$$

$$\approx 2\frac{k_o\eta_o}{4}\left\{\left[t' - j\frac{2}{\pi}t'\left(\ln\left(\frac{\gamma k_o t'}{2}\right)-1\right)\right|_0^{\Delta t_i/2}\right\}$$

$$\approx \frac{k_o\eta_o}{4}\left\{\Delta t_i - j\frac{2\Delta t_i}{\pi}\left(\ln\left(\frac{\gamma k_o \Delta t}{4}\right)-1\right)\right\}$$

$$\boxed{Z_{i,i} \approx \frac{k_o\eta_o\Delta t_i}{4}\left\{1 - j\frac{2}{\pi}\left(\ln\left(\frac{\gamma k_o \Delta t_i}{4e}\right)\right)\right\}}$$

Summary

- The Method of Moment solution for the EFIE for a TMz polarized wave incident on a perfectly electrical conducting cylinder based on a linear segmentation of the contour as well as pulse basis functions and point test functions leads to the discrete linear system of equations:

$$\overline{\overline{Z}}\,\overline{\alpha} = \overline{f}$$

  - where the impedance matrix is computed via:

$$Z_{j,i} = \frac{k_o \eta_o}{4} \int_{t_i}^{t_{i+1}} H_o^{(2)}\left(k_o \left|\vec{r}_j^{\,c} - \vec{r}'(t')\right|\right) dt', \text{ for } j \neq i$$

$$Z_{i,i} \approx \frac{k_o \eta_o \Delta t_i}{4}\left\{1 - j\frac{2}{\pi}\left(\ln\left(\frac{\gamma k_o \Delta t_i}{4e}\right)\right)\right\}, \text{ for } j = i$$

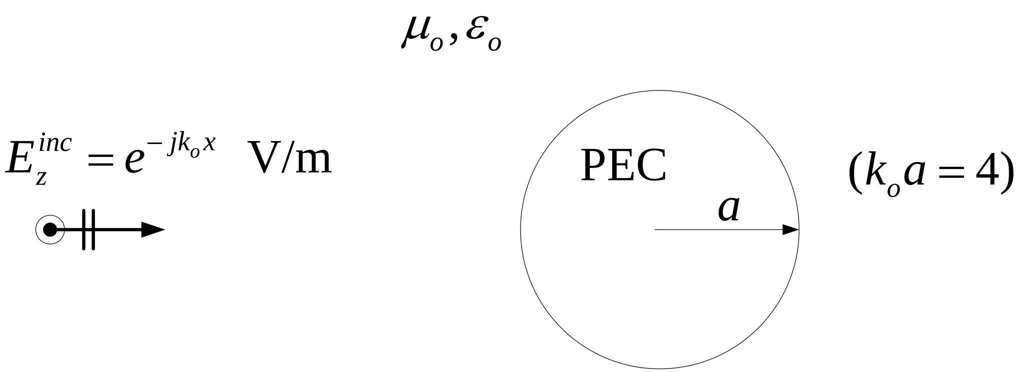
  - with

$$\vec{r}(t') = \vec{r}_i + \left(\vec{r}_{i+1} - \vec{r}_i\right)\frac{t' - t_i}{\Delta t_i}, \quad t' \in t_i, t_{i+1}, \ \Delta t_i = t_{i+1} - t_i$$

  - and:

$$f_j = E_z^{inc}\left(\vec{r}_j^{\,c}\right)$$

Example: Scattering of a TMz Plane Wave by a Circular Cylinder

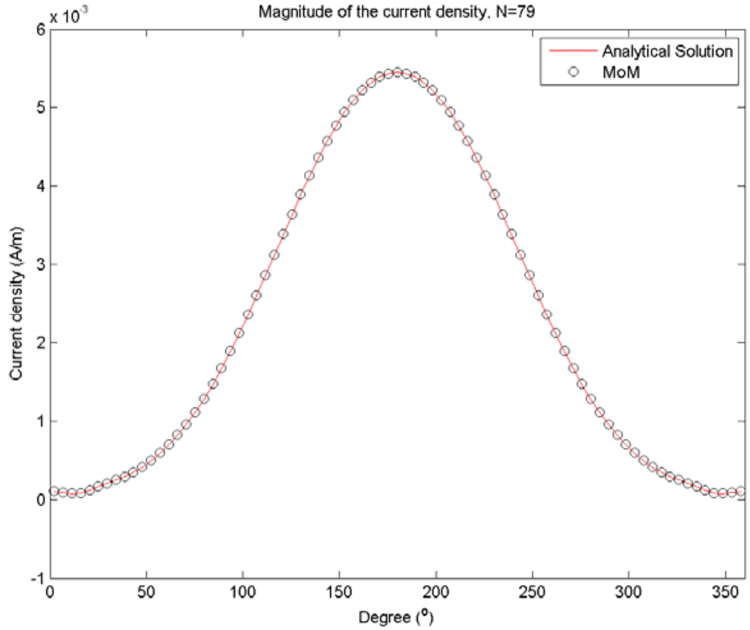


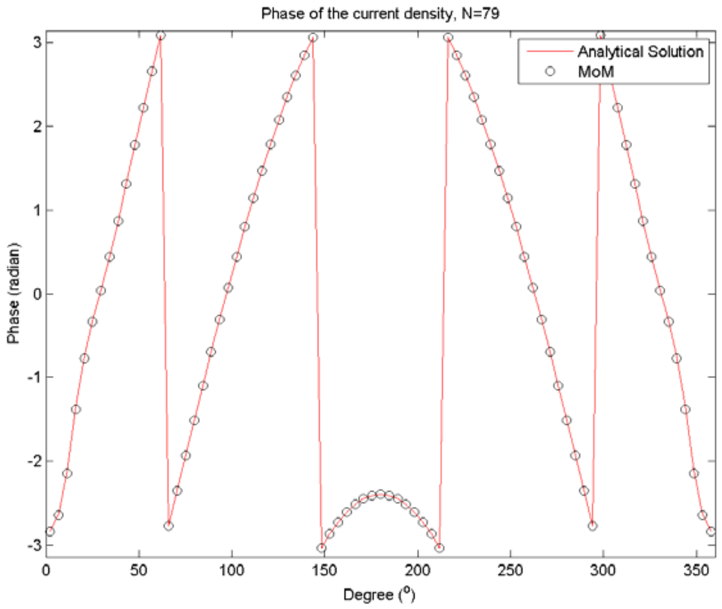
μ_o, ε_o

$E_z^{inc} = e^{-jk_o x}$ V/m

$(k_o a = 4)$

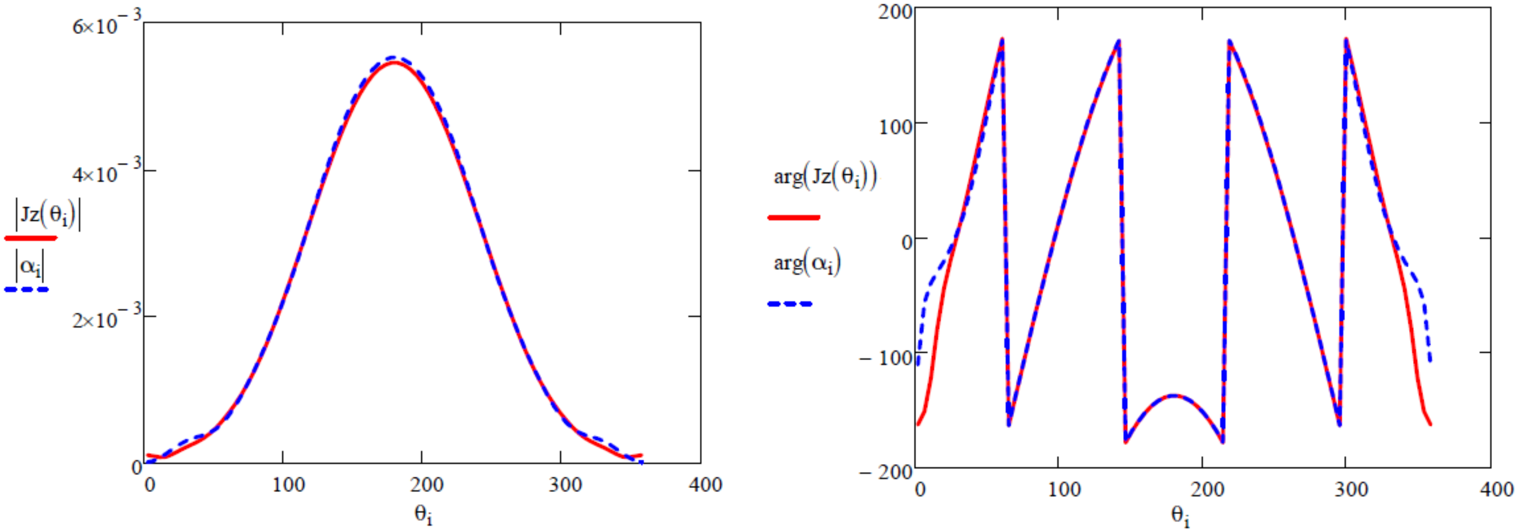
- Piece-wise linear approximation of the surface
  - o EFIE formulation
  - o MoM solution with Pulse basis functions and Point-Matching
  - o Induced current density (using a 3-point integration rule for $Z_{j,i}$), N = 79



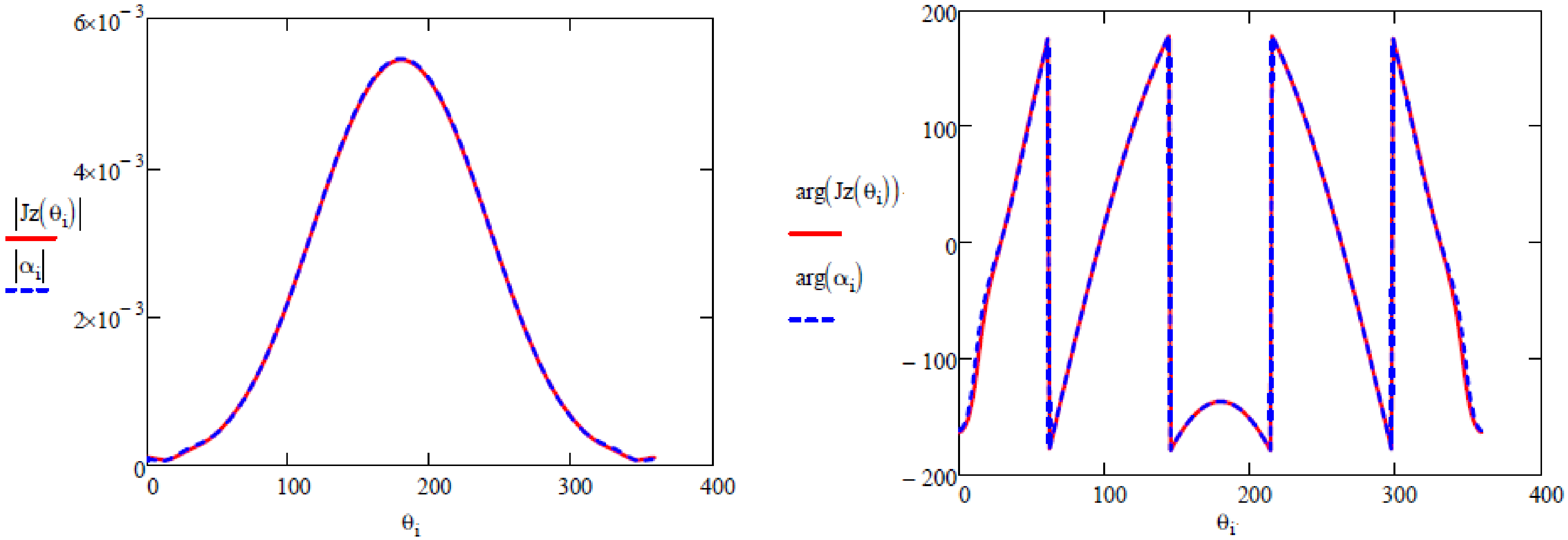


Mid-Point Rule

- N = 80

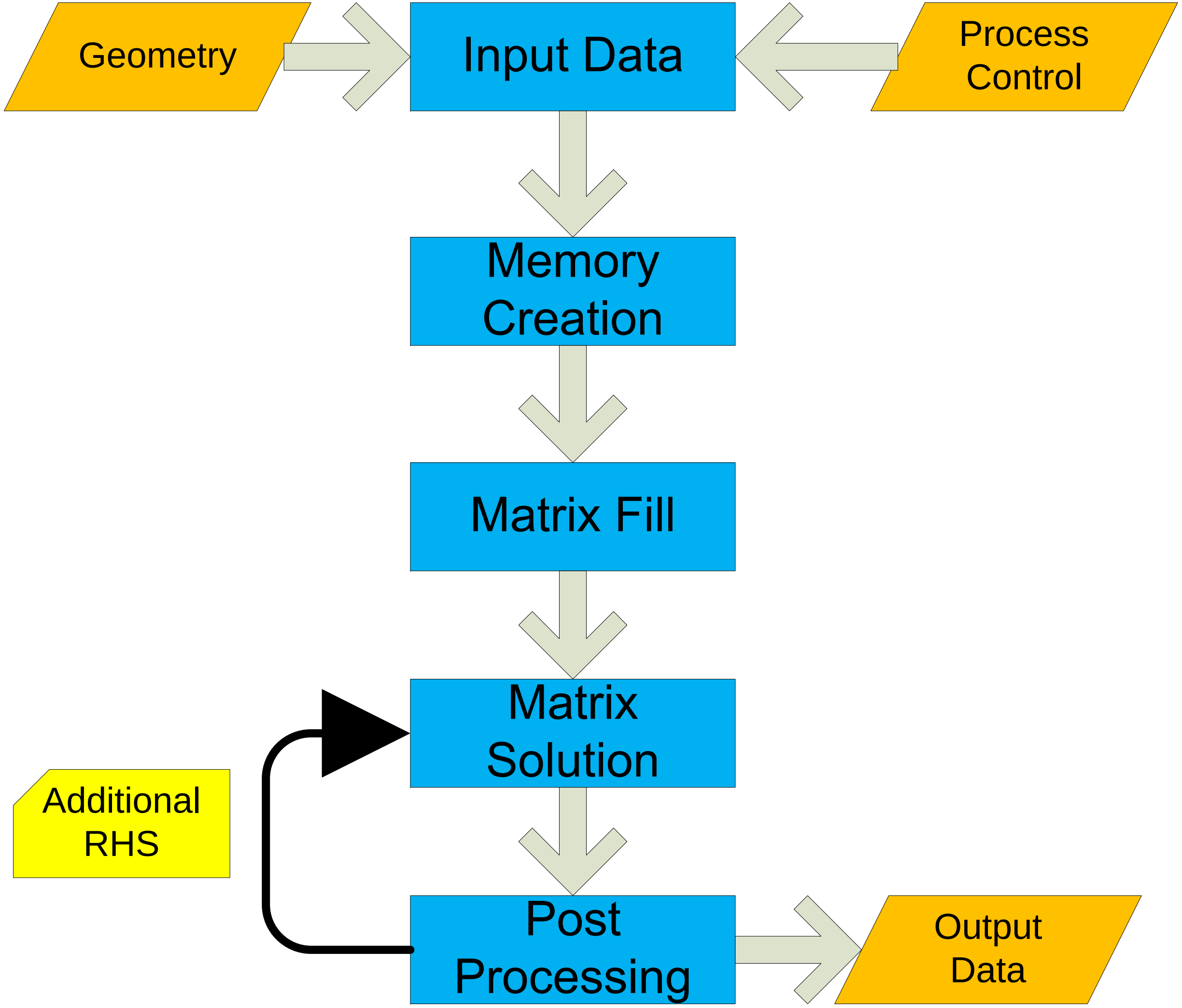


- N = 320



Software Design

- General Top-Level Design:



Geometry

Input Data

Process Control

Memory Creation

Matrix Fill

Matrix Solution

Additional RHS

Post Processing

Output Data

Geometry Data

- Node List
  - Lists the coordinates of the nodes (for 2D problems, don’t need z-coord)

Node #	x-coord	y-coord	z-coord
1	3.765409	-0.034791	1.374978
2	…	…	...

- Edge List
  - lists the bounding nodes defining the edge

Edge #	Node#1	Node#2
1	1	2
2	…	…

Create Memory

- Memory Allocation
  - Dynamically create memory arrays to store:
    - Impedance Matrix
    - RHS vectors (forcing vectors)
    - Other necessary arrays

Fill Operation

- Pre-compute the impedance matrix
  - Double nested loop
    - Outer loop: field cells (*j*)
    - Inner loop: source cells (*i*)
  - Compute the impedance matrix element block for each field-cell/source-cell reaction
    - 1×1 matrix block for the TMz EFIE with pulse basis/point matching
    - $Z_{j,i} \approx \begin{cases} \dfrac{k_o \eta_o \Delta t_i}{4} H_o^{(2)}\left(k_o \left|\vec{r}_j^{\,c} - \vec{r}_i^{\,c}\right|\right), & j \neq i \\ \dfrac{k_o \eta_o \Delta t}{4}\left\{1 - j\dfrac{2}{\pi}\left(\ln\left(\dfrac{\gamma k_o \Delta t}{4e}\right)\right)\right\}, & j = i \end{cases}$

- Pre-compute the forcing vectors (or right-hand-sides):
  - For each excitation, compute the vector:
    - $e_j = E_z\left(\vec{r}\left(t_j^c\right)\right)$
      - which is the incident field evaluated at the midpoint of each segment

Solution Operation

- Solve the linear system of equations
  - Typically done with LU-factorization

$$\overline{\overline{Z}}\,\overline{\alpha} = \overline{e}^{inc}$$

$$\overline{\overline{L}}\,\overline{\overline{U}}\,\overline{\alpha} = \overline{e}^{inc}$$

$$\overline{x} = \overline{\overline{L}}^{-1}\overline{e}^{inc}$$

$$\overline{\alpha} = \overline{\overline{U}}^{-1}\overline{x}$$

  - Very large problems:
    - Direct LU-factorization ($O(N^3)$ operations, $O(N^2)$ memory)
    - Iterative Solutions ($O(N^2)$ operations × # of iterations, $O(N^2)$ memory)
    - Fast Solution Methods ($O(N\log N)$ operations and memory)

Post-Processing Operation

- Computations after computing the solution vector:
  - Equivalent surface currents
  - Near/Far Fields
  - Network Parameters
    - Z/Y-parameters
    - S-parameters
    - Port impedance
    - etc.
  - Power
    - Radiated power
    - Specific absorption
  - etc.