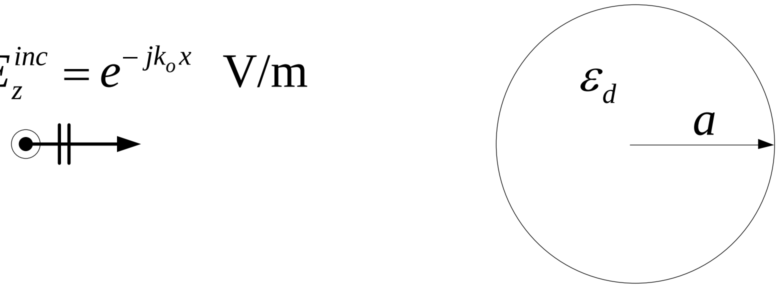
TM_z-Scattering by a Circular Dielectric Cylinder – Analytical Solution

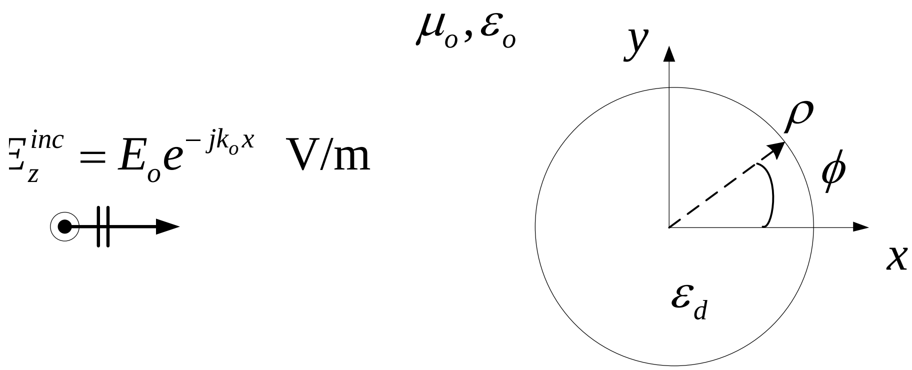
μ_o, ε_o

$$E_z^{inc} = e^{-jk_o x} \text{ V/m}$$



- Consider a uniform plane wave traveling in the positive *x*-direction that is incident on a dielectric cylinder of radius *a* situated in a homogeneous free space.
- Objective:
  - Compute the total fields within the circular dielectric cylinder
  - Compute the total fields outside of the circular dielectric cylinder
- Analytical Solution
  - Based on an addition theorem expansion of the incident field into cylindrical harmonics
  - Posing the scattered field exterior to the cylinder as a superposition of cylindrical harmonics
  - Posing the total field inside to be a superposition of cylindrical harmonics
  - Enforce the boundary conditions on the surface of the cylinder

Cylindrical Harmonic Expansion of the Interior/Exterior Fields



- The incident field:
  - $E_z^{inc} = E_o e^{-jk_o x} = E_o e^{-jk_o \rho \cos\phi} = E_o \sum_{n=0}^{+\infty} j^{-n} \kappa_n J_n(k_o \rho) \cos(n\phi), \qquad \kappa_n = 1 (n=0), = 2 (n \neq 0)$
  - $H_\rho^{inc} = \dfrac{-1}{j\omega\mu_o} \dfrac{1}{\rho} \dfrac{\partial E_z^{inc}}{\partial \phi} = \dfrac{E_o}{jk_o \eta_o} \dfrac{1}{\rho} \sum_{n=0}^{+\infty} j^{-n} n \kappa_n J_n(k_o \rho) \sin(n\phi)$
  - $H_\phi^{inc} = \dfrac{1}{j\omega\mu_o} \dfrac{\partial E_z^{inc}}{\partial \rho} = \dfrac{E_o}{j\eta_o} \sum_{n=0}^{+\infty} j^{-n} \kappa_n J_n'(k_o \rho) \cos(n\phi)$
- Procedure:
  - Pose E_z^{scat} $(\rho > a)$ and E_z^{tot} $(\rho < a)$ as cylindrical harmonic expansion
  - Compute $\vec{H}^{scat}$ $(\rho > a)$ and $\vec{H}^{tot}$ $(\rho < a)$ using Maxwell's equations
  - Solve for unknown coefficients by enforcing boundary conditions and orthogonality of the harmonics.

Cylindrical Harmonic Expansion of the Scat/Tot Fields

- Scattered field ($\rho > a$):
  - $E_z^{scat} = E_o \sum_{n=0}^{+\infty} a_n j^{-n} \kappa_n H_n^{(2)}\left(k_o \rho\right) \cos(n\phi),\ \rho > a$
  - $H_\phi^{scat} = \frac{E_o}{j\eta_o} \sum_{n=0}^{+\infty} a_n j^{-n} \kappa_n H_n^{(2)\prime}\left(k_o \rho\right) \cos(n\phi)$
- Total field ($\rho < a$):
  - $E_z^{tot} = E_o \sum_{n=0}^{+\infty} b_n j^{-n} \kappa_n J_n\left(k_d \rho\right) \cos(n\phi),\ \rho < a$
  - $H_\phi^{tot} = \frac{E_o}{j\eta_d} \sum_{n=0}^{+\infty} b_n j^{-n} \kappa_n J_n{}'\left(k_d \rho\right) \cos(n\phi)$
    - where $k_d = \omega\sqrt{\varepsilon_d \mu_o}$ and $\eta_d = \sqrt{\frac{\mu_o}{\varepsilon_d}}$
- a_n and b_n are unkowns.
  - How do we solve for them?

Solving for the Unknown Coefficients

- Boundary conditions:
  - $E_z^{inc}\left(\rho=a^+\right)+E_z^{scat}\left(\rho=a^+\right)=E_z^{tot}\left(\rho=a^-\right)$
  - $H_\phi^{inc}\left(\rho=a^+\right)+H_\phi^{scat}\left(\rho=a^+\right)=H_\phi^{tot}\left(\rho=a^-\right)$
- Applying orthogonality, this leads to:
  - $E_o j^{-n}\kappa_n J_n\left(k_o a\right)\cos(n\phi)+E_o a_n j^{-n}\kappa_n H_n^{(2)}\left(k_o a\right)\cos(n\phi)=E_o b_n j^{-n}\kappa_n J_n\left(k_d a\right)\cos(n\phi)$
  - $\frac{E_o}{j\eta_o} j^{-n}\kappa_n J_n'\left(k_o a\right)\cos(n\phi)+\frac{E_o}{j\eta_o} a_n j^{-n}\kappa_n H_n^{(2)\prime}\left(k_o a\right)\cos(n\phi)=\frac{E_o}{j\eta_o} b_n j^{-n}\kappa_n J_n'\left(k_d a\right)\cos(n\phi)$
- The 2×2 can be solved for a_n and b_n:
  - $a_n=\dfrac{\eta_o J_n'\left(k_d a\right)J_n\left(k_o a\right)-\eta_d J_n'\left(k_o a\right)J_n\left(k_d a\right)}{\eta_d J_n\left(k_d a\right)H_n^{(2)\prime}\left(k_o a\right)-\eta_o J_n'\left(k_d a\right)H_n^{(2)}\left(k_o a\right)}$
  - $b_n=\eta_d\dfrac{J_n\left(k_o a\right)H_n^{(2)\prime}\left(k_o a\right)-J_n'\left(k_o a\right)H_n^{(2)}\left(k_o a\right)}{\eta_d J_n\left(k_d a\right)H_n^{(2)\prime}\left(k_o a\right)-\eta_o J_n'\left(k_d a\right)H_n^{(2)}\left(k_o a\right)}$

Ez-total

- The total electric field E_z^{tot} can be written as:
  - $$E_z^{tot} = \begin{cases} E_o \sum_{n=0}^{+\infty} b_n j^{-n} \kappa_n J_n\left(k_d \rho\right)\cos(n\phi), & \rho \le a \\ E_o \sum_{n=0}^{+\infty} j^{-n} \kappa_n \left[J_n\left(k_o \rho\right) + a_n H_n^{(2)}\left(k_o \rho\right)\right]\cos(n\phi), & \rho \ge a \end{cases}$$
    - Note, $E_z^{tot}\left(\rho = a^+, \phi\right) = E_z^{tot}\left(\rho = a^-, \phi\right) = E_z^{tot}\left(\rho = a, \phi\right)$

Computing the Far Field

- Often it is desirable to compute the "far" scattered field.
  - Region where the distance from the cylinder $(\rho - a)/\lambda >> 1$
  - Also quantified as the region where $k_o\rho >> 1$

- Asymptotic approximation of the Hankel function:
  - $\lim\limits_{k_o\rho\to\infty} H_n^{(2)}(k_o\rho) \approx \sqrt{\frac{2j}{\pi k_o\rho}}\, j^n e^{-jk_o\rho}$
- Therefore, in the far field:
  - $$\lim_{k_o\rho\to\infty} E_z^{scat}(\rho,\phi) = \lim_{k_o\rho\to\infty} -E_o \sum_{n=0}^{+\infty} j^n (-1)^n \kappa_n a_n H_n^{(2)}(k_o\rho)\cos(n\phi)$$
    $$\approx -E_o \sqrt{\frac{2j}{\pi k_o\rho}} e^{-jk_o\rho} \sum_{n=0}^{+\infty} j^{2n} (-1)^n \kappa_n a_n \cos(n\phi)$$
- Therefore,
  - $E_z^{scat-ff}(\rho,\phi) = -E_o \sqrt{\frac{2j}{\pi k_o\rho}} e^{-jk_o\rho} \sum\limits_{n=0}^{+\infty} \kappa_n a_n \cos(n\phi)$

Echo Width

- The echo width is the equivalent area proportional to the apparent size of the target that scatters the field in a given direction relative to the power incident from another direction.
  - $$\sigma_{2D}(\phi) = \lim_{\rho\to\infty}(2\pi\rho\Delta z)\frac{\text{Scattered Power Density}}{\text{Incident Power Density}} = \lim_{\rho\to\infty} 2\pi\rho \frac{\left|E_z^{scat}\right|^2 / \eta_o}{\left|E_z^{inc}\right|^2 / \eta_o}$$
- In the far field:
  - $$\left|E_z^{scat-ff}(\rho,\phi)\right|^2 = E_o^2 \frac{2}{\pi k_o \rho}\left|\sum_{n=0}^{+\infty}\kappa_n a_n \cos(n\phi)\right|^2$$
- Therefore
  - $$\sigma_{2D}(\phi) = \lim_{\rho\to\infty} 2\pi\rho \frac{E_o^2 \dfrac{2}{\pi k_o \rho}\left|\sum_{n=0}^{+\infty}\kappa_n a_n \cos(n\phi)\right|^2 / \eta_o}{E_o^2 / \eta_o}$$
- or
  - $$\boxed{\sigma_{2D}(\phi) = \frac{4}{k_o}\left|\sum_{n=0}^{+\infty}\kappa_n a_n \cos(n\phi)\right|^2}$$

MathCad Example

$ka = 4 \qquad \eta = 376.73031346177066 \qquad \varepsilon r = 4 \qquad \eta d := \frac{\eta}{\sqrt{\varepsilon r}} \qquad kda := ka \cdot \sqrt{\varepsilon r}$

$$an(n) := \frac{\eta \cdot Jnp(n,kda) \cdot Jn(n,ka) - \eta d \cdot Jnp(n,ka) \cdot Jn(n,kda)}{\eta d \cdot Jn(n,kda) \cdot Hn2p(n,ka) - \eta \cdot Jnp(n,kda) \cdot Hn2(n,ka)}$$

$$\sigma 2dEx(\phi) := 10\log\left[\frac{4}{ko} \cdot \left[\left|\sum_{n=0}^{Nmax}\left(\kappa(n) \cdot an(n) \cdot \cos\left(n \cdot \phi \cdot \frac{\pi}{180}\right)\right)\right|\right]^2\right]$$