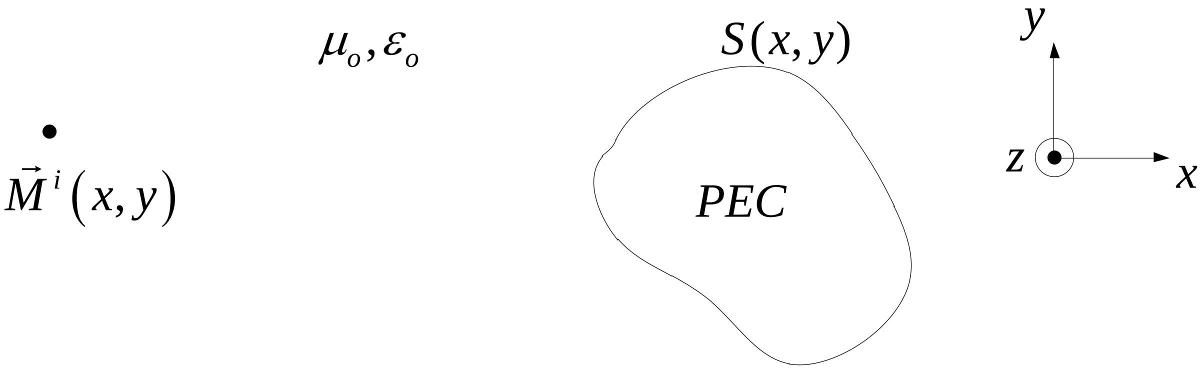
TE_z-Scattering by a PEC Cylinder – MFIE Solution



- Consider a magnetic current density $\vec{M}^i(x,y) = \hat{z}M_z^i$ that is infinitely long and invariant w.r.t. the *z*-axis radiating in the presence of an infinitely long PEC cylinder defined by the surface $S(x,y)$ which is situated in an infinite, homogeneous, unbounded media.
- Objective:
  - Compute the currents induced on the PEC cylinder
  - Compute the field scattered by the PEC Cylinder
- Solution
  - Pose the equivalent problem
  - Establish the Magnetic Field Integral Equation for this problem
  - Solve for the induced currents via the Method of Weighted Residuals
  - Compute the scattered electric field from the induced currents

Polarization

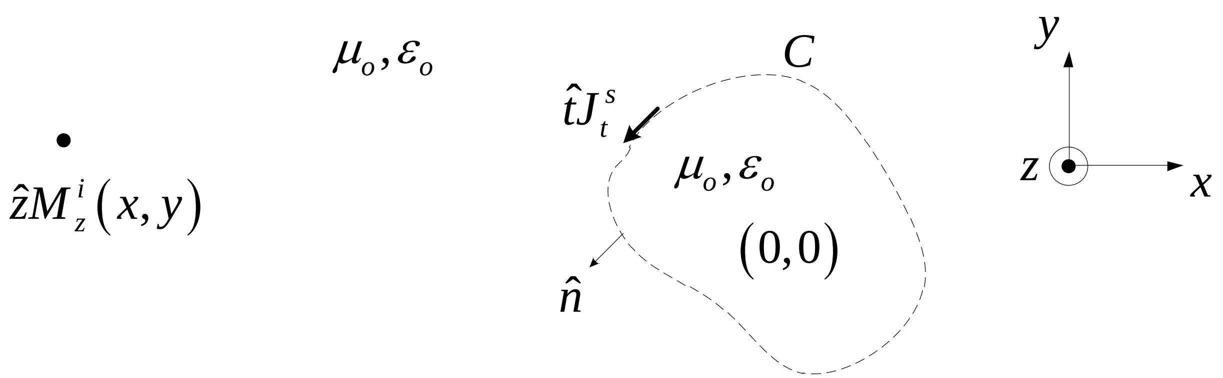
- Note that $\vec{M}^i$ is *z*-directed, and infinite and invariant along *z*.
  - $\vec{M}^i = \hat{z} M_z^i \left(x, y\right)$
- From vector potential theory:
  - $\vec{F} = \hat{z} \psi_e \left(x, y\right)$
- Therefore, the field radiated by the current density is TE_z polarized.
  - Thus:

$$\boxed{\begin{array}{ll} E_x = -\dfrac{\partial}{\partial y}\psi_e & H_x = 0 \\ E_y = \dfrac{\partial}{\partial x}\psi_e & H_y = 0 \\ E_z = 0 & H_z = -j\omega e \psi_e + \dfrac{1}{j\omega\mu}\dfrac{\partial^2}{\partial z^2}\psi_e \end{array}}$$

- Since the cylinder is also infinite and invariant along the *z*-direction, the scattered field is TE_z polarized.
- Also, since *E* is transverse only, and *z*-invariant, then only an *x*,*y*-directed current will be induced on the surface of the PEC cylinder (i.e., $J_{z_s} = 0$)

The Equivalent Problem

$$\left(\vec{E}^{inc}+\vec{E}^{scat},\vec{H}^{inc}+\vec{H}^{scat}\right)$$



- Following Green's second identity, we can place an equivalent current density

$$\vec{J}_s=\hat{n}\times\vec{H}^{tot}\Big|_C=\hat{t}J_t^s$$

on *S* that is effectively radiating in a homogeneous free space.

- The equivalent current radiates the scattered field in the region exterior to *C*, and the negative of the incident field in the region interior to *C*.
- The magnetic field integral equation (MFIE) is derived by enforcing the physical boundary condition of the tangential magnetic field on the surface of the PEC:

$$\hat{n}\times\vec{H}^{tot}\Big|_S=\vec{J}_s,$$

or,

$$\hat{n}\times\vec{H}^{inc}\Big|_C=\vec{J}_s-\hat{n}\times\vec{H}^{scat}\Big|_C$$

- The scattered field is computed as:

$$\begin{aligned}\vec{H}^{scat} &= \nabla \times \vec{A} \\ &= \int_C \nabla \times \frac{1}{4j} H_0^{(2)}\left(k_o\left|\vec{r}-r'\right|\right)\vec{J}_s d\ell'\end{aligned}$$

- Because of the jump discontinuity of $\hat{n}\times\vec{H}^{scat}$ across *C*, the integral is dual valued.
  - We must apply a principal value integral to evaluation this integral (see notes)
- After applying the principal value, we find that:

$$\hat{n}\times\vec{H}^{scat} = \frac{1}{2}\vec{J}_s + \hat{n}\times\fint_C \nabla \times \frac{1}{4j} H_0^{(2)}\left(k_o\left|\vec{r}-r'\right|\right)\vec{J}_s\left(\vec{r}'\right)d\ell'$$

  - where $\frac{1}{2}\vec{J}_s$ is the residual and $\fint_C$ is the principal value.
- As a consequence:

$$\hat{n}\times\vec{H}^{inc}\Big|_C = \frac{1}{2}\vec{J}_s - \hat{n}\times\fint_C \nabla \times \frac{1}{4j} H_0^{(2)}\left(k_o\left|\vec{r}-r'\right|\right)\vec{J}_s\left(\vec{r}'\right)d\ell'$$

The MFIE

- Applying the vector identity: $\nabla\times\phi\vec{A}=\phi\nabla\times\vec{A}+\nabla\phi\times\vec{A}$, and $\nabla\times\vec{J}_s\left(\vec{r}'\right)=0$
  - Because the curl is in the unprimed coordinate space (field-space).
  - The MFIE becomes:

$$\hat{n}\times\vec{H}^{inc}\Big|_C=\frac{1}{2}\vec{J}_s-\frac{1}{4j}\hat{n}\times\oint_C\nabla H_0^{(2)}\left(k_o\left|\vec{r}-r'\right|\right)\times\vec{J}_s\left(\vec{r}'\right)d\ell'$$

- Define
  - $\vec{R}=\vec{r}-\vec{r}'$ = the radial vector between the source and field points
  - $R=\left|\vec{r}-\vec{r}'\right|=\left|\vec{R}\right|$ = radial distance between the source and field points
- Then, using cylindrical coordinates:
  - $\nabla H_0^{(2)}\left(k_oR\right)=\dfrac{\vec{R}}{R}\dfrac{\partial}{\partial R}H_0^{(2)}\left(k_oR\right)=-k_o\dfrac{\vec{R}}{R}H_1^{(2)}\left(k_oR\right)$
- Finally, the MFIE is written as:

$$\boxed{\hat{n}\times\vec{H}^{inc}\left(\vec{r}\right)\Big|_C=\frac{1}{2}\vec{J}_s\left(\vec{r}\right)+\frac{k_o}{4j}\hat{n}\times\oint_C\frac{\vec{R}}{R}\times\vec{J}_s\left(\vec{r}'\right)H_1^{(2)}\left(k_oR\right)d\ell'}$$

  - where the field point $\vec{r}\in C$ and $\hat{n}$ is the outward normal evaluated at $\vec{r}$

The Method of Weighted Residuals Solution

$$\boxed{\int_C \vec{T}(\vec{r})\cdot\hat{n}\times\vec{H}^{inc}(\vec{r})\,d\ell = \frac{1}{2}\int_C \vec{T}(\vec{r})\cdot\vec{J}_s(\vec{r})\,d\ell + \frac{k_o}{4j}\int_C \vec{T}(\vec{r})\cdot\hat{n}\times\fint_C \frac{\vec{R}}{R}\times\vec{J}_s(\vec{r}\,')H_1^{(2)}(k_oR)\,d\ell' \;\; d\ell}$$

- We can express this in the operator form:

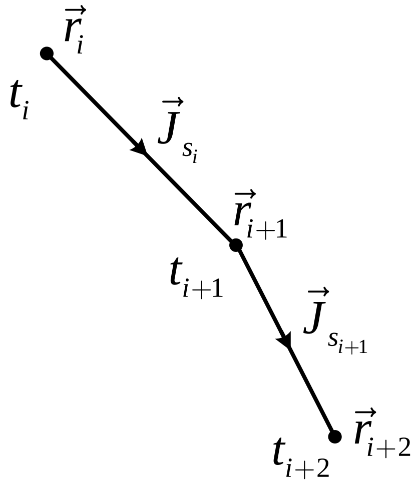
$$f = \mathcal{L}x$$

- Recall, to solve this via the method of weighted residuals:
  - Expand the current density via a *basis function* space $\{\phi_i\}$ with support on *C* that spans a polynomial space of order *p* defined on *C.*
  - This is done here in parametric coordinates:

$$\vec{J}_t^{\,s}(l) = \sum_{i=1}^{N}\alpha_i\vec{\phi}_i(l)$$

    - The basis function space interpolates *J* over *C*
    - This time, the basis functions are also vectors!
      - The vector current must be tangential to the contour
  - Test functions must also be chosen. Again, they are vectors.
    - Due to the cross product with the normal, the test vectors must also be tangential to the contour, and in the transverse direction.

Discretization



- The outer contour of the PEC is discretized with linear segments
- Current density is assumed to be tangent to the contour
  - *i*-th segment: $\hat{t}_i = \dfrac{\vec{r}_{i+1} - \vec{r}_i}{\left|\vec{r}_{i+1} - \vec{r}_i\right|} = \dfrac{\vec{r}_{i+1} - \vec{r}_i}{t_{i+1} - t_i} = \dfrac{\vec{r}_{i+1} - \vec{r}_i}{\Delta t_i}$
- Basis function expansion:
  - $\vec{J}_s \approx \sum_{i=1}^{N} \alpha_i P\left(t; t_i, t_{i+1}\right) \hat{t}_i$
    - Pulse function: $P\left(t; t_i, t_{i+1}\right) = \begin{cases} 1, & t_i < t < t_{i+1} \\ 0, & \text{else} \end{cases}$
    - α_i are the unknown coefficients
- Test functions
  - $\vec{T}_j = \hat{t}_j \delta\left(\left|\vec{r} - r_j^c\right|\right)$
    - Delta function at cell centers, tangent to cell

The Discrete MFIE

$$\hat{t}_j \cdot \hat{n}_j \times \vec{H}^{inc}\left(\vec{r}_j^{\,c}\right) = \frac{1}{2}\hat{t}_j \cdot \hat{t}_j \alpha_j + \frac{k_o}{4j}\hat{t}_j \cdot \hat{n}_j \times \sum_{i=1}^{N} \alpha_i \int_{t_i}^{t_{i+1}} \frac{\vec{R}_j}{R_j} \times \hat{t}_i H_1^{(2)}\left(k_o R_j\right) d\ell'$$

- where
  - $\vec{R}_j = \vec{r}_j^{\,c} - \vec{r}\,'$, & $R_j = \left|\vec{R}_j\right|$
- This can be expressed now as a linear system of equations:

$$\bar{f}^{\,h} = \bar{\bar{Y}}\bar{\alpha}$$

- where, the *j*-th entry of the forcing vector is:

$$f_j^h = \hat{t}_j \cdot \hat{n}_j \times \vec{H}^{inc}\left(\vec{r}_j^{\,c}\right)$$

- and the *j*,*i*-th entry of the system matrix is:,

$$Y_{j,i} = \frac{k_o}{4j}\hat{t}_j \cdot \hat{n}_j \times \int_{t_i}^{t_{i+1}} \frac{\vec{R}_j}{R_j} \times \hat{t}_i H_1^{(2)}\left(k_o R_j\right) d\ell' \;\; ,(j \neq i)$$

- and the *j*,*j*-th diagonal entry is:

$$Y_{j,j} = \frac{1}{2} + \frac{k_o}{4j}\hat{t}_j \cdot \hat{n}_j \times \int_{t_j}^{t_{j+1}} \frac{\vec{R}_j}{R_j} \times \hat{t}_j H_1^{(2)}\left(k_o R_j\right) d\ell'$$

- The approximate solution for the surface currents is derived via:

$$\bar{\alpha} = \bar{\bar{Y}}^{-1}\bar{f}^{\,h}$$

Evaluating the System Matrix

$$Y_{j,i} = \frac{k_o}{4j}\hat{t}_j \cdot \hat{n}_j \times \int_{t_i}^{t_{i+1}} \frac{\vec{R}_j}{R_j} \times \hat{t}_i H_1^{(2)}\left(k_o R_j\right) d\ell' \;\; ,(j \neq i)$$

- Approximate with a mid-point rule:

$$Y_{j,i} \approx \Delta t_i \frac{k_o}{4j}\hat{t}_j \cdot \hat{n}_j \times \hat{R}_{j,i} \times \hat{t}_i H_1^{(2)}\left(k_o R_{j,i}\right) \;\; ,(j \neq i)$$

  - where $\vec{R}_{j,i} = \vec{r}_j^{\,c} - \vec{r}_i^{\,c}$, $R_{j,i} = \left|\vec{R}_{j,i}\right|$, and $\hat{R}_{j,i} = \dfrac{\vec{R}_{j,i}}{R_{j,i}}$

- Note that:
  - $\hat{t}_j \cdot \hat{n}_j \times \hat{R}_{j,i} \times \hat{t}_i = \left(\hat{t}_j \cdot \hat{R}_{j,i}\right)\left(\hat{n}_j \cdot \hat{t}_i\right) - \left(\hat{t}_j \cdot \hat{t}_i\right)\left(\hat{R}_{j,i} \cdot \hat{n}_j\right) = \gamma_{j,i}$

- Therefore,

$$Y_{j,i} \approx \left[\frac{k_o}{4j}\Delta t_i \, \gamma_{j,i} \, H_1^{(2)}\left(k_o R_{j,i}\right)\right] \;,(j \neq i)$$

Computing the Self Term

$$Y_{j,j} = \frac{1}{2} + \frac{k_o}{4j}\hat{t}_j \cdot \hat{n}_j \times \int_{t_j}^{t_{j+1}} \hat{R}_i \times \hat{t}_j H_1^{(2)}\left(k_o R_j\right) d\ell'$$

- For the self-term, it is noted that:
  - $\gamma_{j,j} = \hat{t}_j \cdot \hat{n}_j \times \hat{R}_j \times \hat{t}_i = \left(\hat{t}_j \cdot \hat{R}_j\right)\left(\hat{n}_j \cdot \hat{t}_j\right) - \left(\hat{t}_j \cdot \hat{t}_j\right)\left(\hat{R}_j \cdot \hat{n}_j\right)$
- Observe that:
  - $\left(\hat{n}_j \cdot \hat{t}_j\right) \equiv 0$.
  - Also, $\hat{R}_j$ is tangential to the cell. Therefore, $\left(\hat{R}_j \cdot \hat{n}_j\right) \equiv 0$
  - Also, since $\int_{t_j}^{t_{j+1}} H_1^{(2)}\left(k_o R_j\right) d\ell'$ is bounded, we find that:
    - $\frac{k_o}{4j}\hat{t}_j \cdot \hat{n}_j \times \int_{t_j}^{t_{j+1}} \hat{R}_i \times \hat{t}_j H_1^{(2)}\left(k_o R_j\right) d\ell' \equiv 0$
- Therefore,

$$Y_{j,j} = \frac{1}{2}$$

Summary:

- System matrix:

$$Y_{j,i} \approx \begin{cases} \left[\dfrac{k_o}{4j} \Delta t_i \, \gamma_{j,i} \, H_1^{(2)}\left(k_o R_{j,i}\right) \right], & (j \neq i) \\ \dfrac{1}{2}, & (j = i) \end{cases}$$

- Forcing vector:

$$f_j^h = \hat{t}_j \cdot \hat{n}_j \times \vec{H}^{inc}\left(\vec{r}_j^{\,c}\right)$$

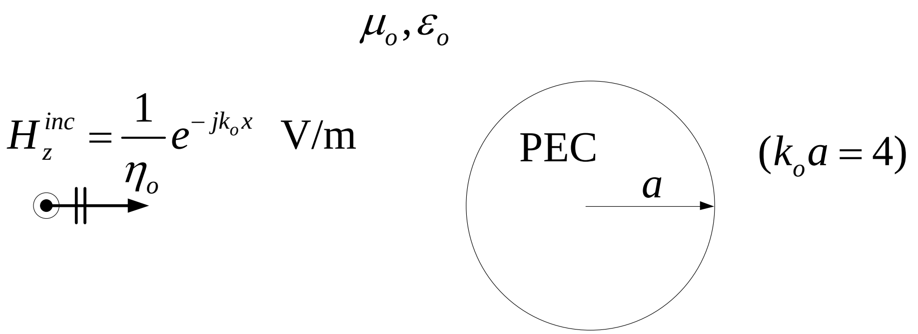
- Solution vector:

$$\bar{\alpha} = \bar{\bar{Y}}^{-1} \bar{f}^h$$

Logistics

- Position vector $\vec{r}_i = x_i\hat{x} + y_i\hat{y}$
- Segement length:
  - $\Delta t_i = \left|\vec{r}_{i+1} - \vec{r}_i\right| = \sqrt{(x_{i+1}-x_i)^2 + (y_{i+1}-y_i)^2}$
- Tangent vector:
  - $\hat{t}_i = \dfrac{\vec{r}_{i+1} - \vec{r}_i}{\left|\vec{r}_{i+1} - \vec{r}_i\right|} = \dfrac{\left(x_{i+1}-x_i\right)\hat{x} + \left(y_{i+1}-y_i\right)\hat{y}}{\Delta t_i}$
- Normal vector:
  - Normal vector requires knowledge of the segment orientation
  - $\hat{n}_i = \hat{z}\times\hat{t}_i = \left(-\left(y_{i+1}-y_i\right)\hat{x} + \left(x_{i+1}-x_i\right)\hat{y}\right)/\,\Delta t_i$
    - This is based on the assumption that the segments are listed in a clockwise direction.
    - Segments must be consistently oriented.

- **Example: Scattering of a TEz Plane Wave by a Circular Cylinder**

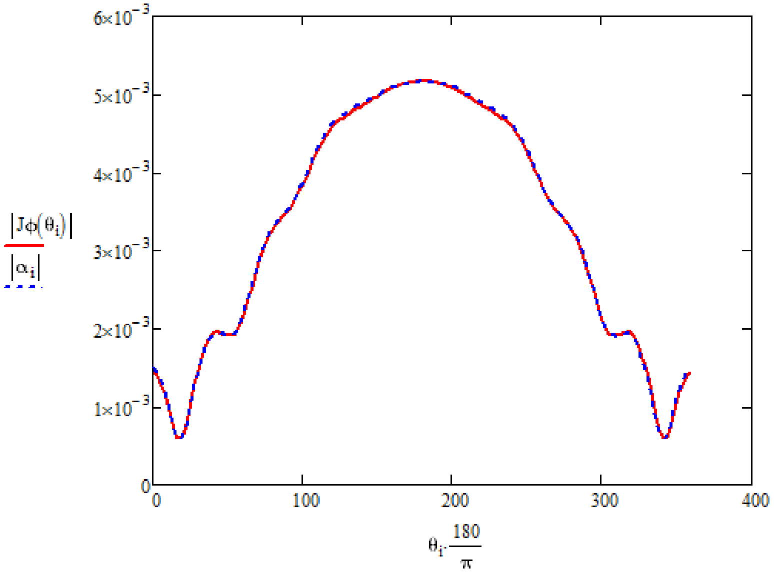


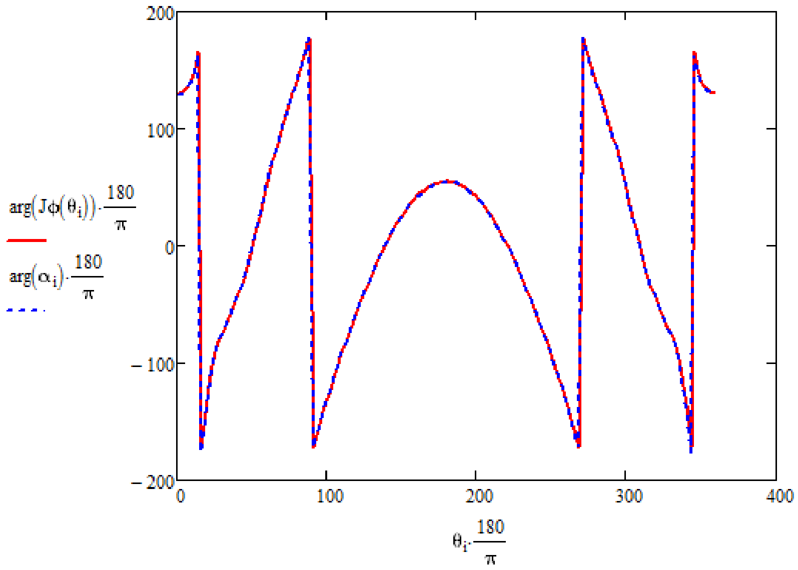
μ_o, ε_o

$$H_z^{inc} = \frac{1}{\eta_o} e^{-jk_o x} \text{ V/m}$$

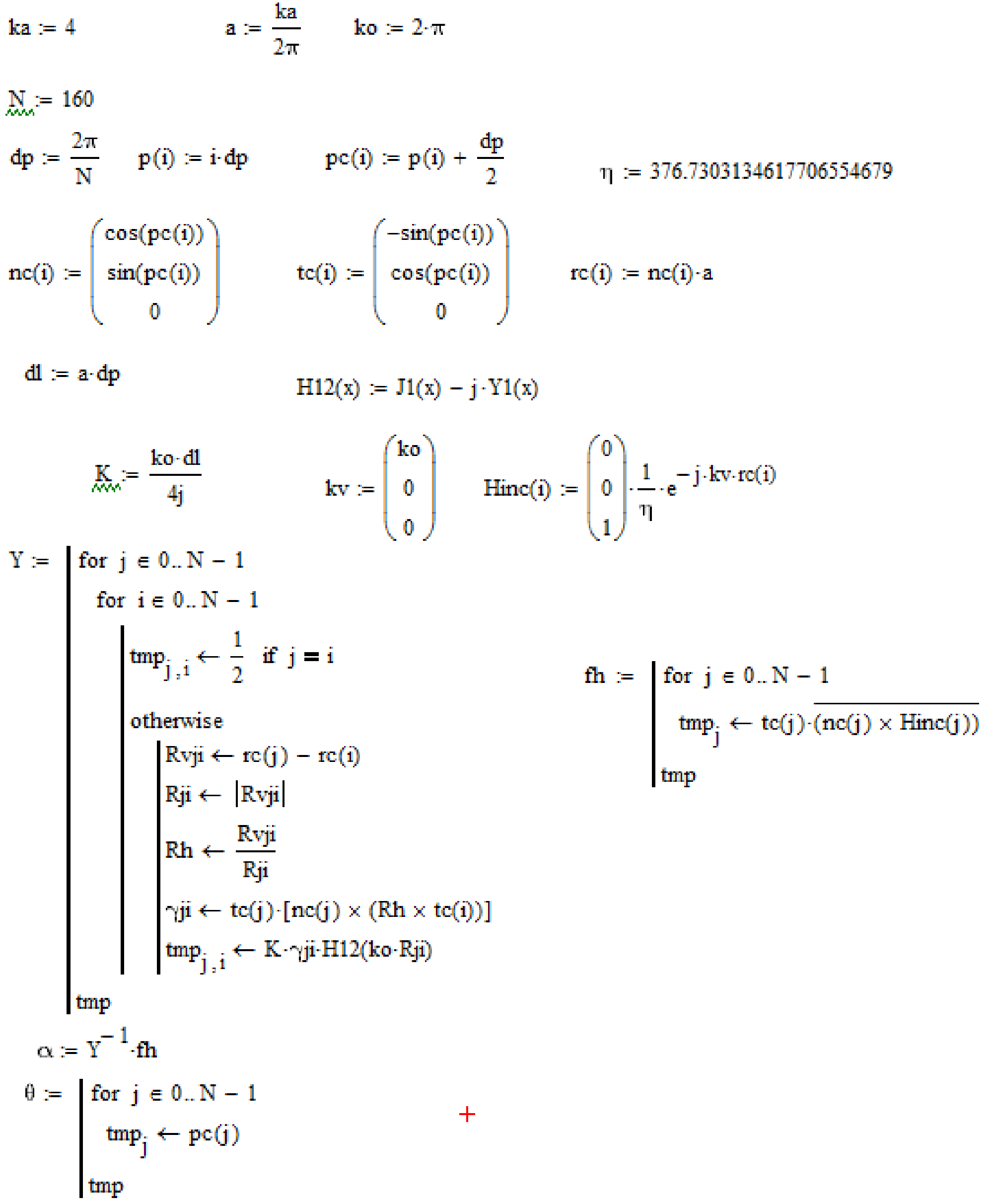
$(k_o a = 4)$

- Piece-wise linear approximation of the surface
    - o MFIE formulation
    - o MoM solution with Pulse basis functions and Point-Matching
    - o Induced current density (Exact (J_ϕ), vs. MoM (α))
        - ▪ 160 uniformly spaced segments





MathCad Code



$ka := 4 \qquad a := \frac{ka}{2\pi} \qquad ko := 2\cdot\pi$

$N := 160$

$dp := \frac{2\pi}{N} \qquad p(i) := i\cdot dp \qquad pc(i) := p(i) + \frac{dp}{2} \qquad \eta := 376.7303134617706554679$

$nc(i) := \begin{pmatrix} \cos(pc(i)) \\ \sin(pc(i)) \\ 0 \end{pmatrix} \qquad tc(i) := \begin{pmatrix} -\sin(pc(i)) \\ \cos(pc(i)) \\ 0 \end{pmatrix} \qquad rc(i) := nc(i)\cdot a$

$dl := a\cdot dp \qquad H12(x) := J1(x) - j\cdot Y1(x)$

$K := \frac{ko\cdot dl}{4j} \qquad kv := \begin{pmatrix} ko \\ 0 \\ 0 \end{pmatrix} \qquad Hinc(i) := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\cdot\frac{1}{\eta}\cdot e^{-j\cdot kv\cdot rc(i)}$

```
Y := | for j ∈ 0..N − 1
     |   for i ∈ 0..N − 1
     |     | $tmp_{j,i} \leftarrow \frac{1}{2}$  if j = i
     |     | otherwise
     |     |   | $Rvji \leftarrow rc(j) - rc(i)$
     |     |   | $Rji \leftarrow |Rvji|$
     |     |   | $Rh \leftarrow \frac{Rvji}{Rji}$
     |     |   | $\gamma ji \leftarrow tc(j)\cdot[nc(j) \times (Rh \times tc(i))]$
     |     |   | $tmp_{j,i} \leftarrow K\cdot\gamma ji\cdot H12(ko\cdot Rji)$
     | tmp
```

```
fh := | for j ∈ 0..N − 1
      |   $tmp_j \leftarrow tc(j)\cdot\overline{(nc(j) \times Hinc(j))}$
      | tmp
```

$\alpha := Y^{-1}\cdot fh$

```
θ := | for j ∈ 0..N − 1
     |   $tmp_j \leftarrow pc(j)$
     | tmp
```

Scattered Fields:

- Once the method of moment solution is computed, the scattered field can be computed from the induced current:

$$\vec{H}^{scat}(\vec{r}) = \hat{z}\cdot\nabla\times\vec{A}(\vec{r}) = \hat{z}\cdot\int_C \nabla\times\frac{1}{4j}H_0^{(2)}\left(k_o\left|\vec{r}-r'\right|\right)\vec{J}_s(\vec{r}\,')\,d\ell'$$

$$= \frac{-k_o}{4j}\,\hat{z}\cdot\int_C \frac{\vec{R}}{R}\times\vec{J}_s(\vec{r}\,')H_1^{(2)}(k_oR)\,d\ell'$$

  - From the discrete currents:

$$\vec{H}^{scat}(\vec{r}) = \frac{-k_o}{4j}\,\hat{z}\cdot\sum_{i=1}^{N}\alpha_i\int_{t_i}^{t_{i+1}}\frac{\vec{R}}{R}\times\hat{t}_iH_1^{(2)}(k_oR)\,dt'$$

Far Field Approximation

$$\vec{H}^{scat}(\vec{r}) = \frac{-k_o}{4j}\,\hat{z}\cdot\int_C \frac{\vec{R}}{R}\times\vec{J}_s(\vec{r}')\,H_1^{(2)}(k_o R)\,d\ell'$$

- Quite often, one is interested in the scattered field in the *far-field zone*
  - o where $\max(R) >> \lambda$
- In this region, we can use a far-field approximation of $H_1^{(2)}(k_o R)$
- Identity:

$$\lim_{x\to\infty} H_n^{(2)}(x) \approx \sqrt{\frac{2}{\pi x}}\,e^{-j\left[x - n\frac{\pi}{2} - \frac{\pi}{4}\right]}$$

- Therefore,

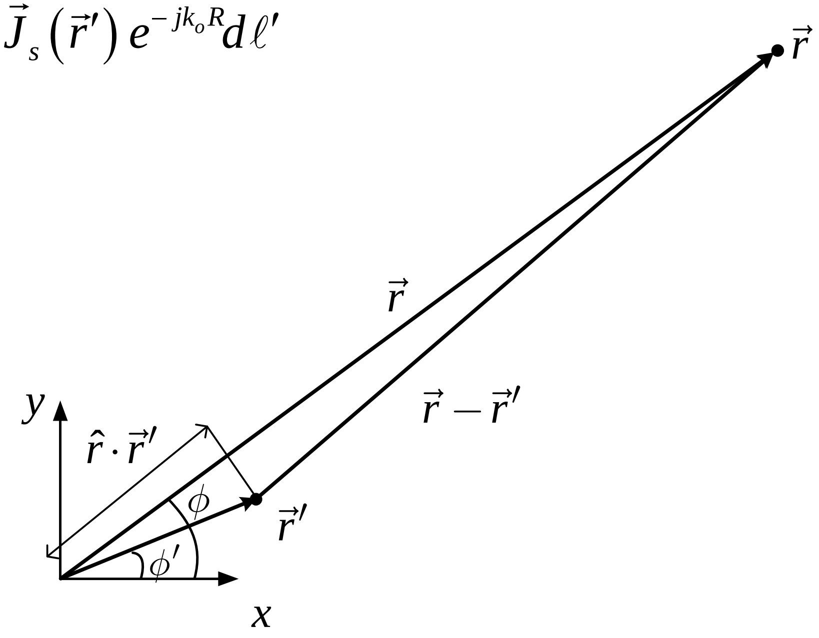
$$\lim_{x\to\infty} H_1^{(2)}(k_o R) \approx \sqrt{\frac{2}{\pi k_o R j}}\,e^{-jk_o R}$$

  - o Consequently:

$$H_z^{scat\,ff}(\vec{r}) \approx \frac{-k_o}{4j}\,\hat{z}\cdot\int_C \hat{R}\times\vec{J}_s(\vec{r}')\sqrt{\frac{2}{\pi k_o R j}}\,e^{-jk_o R}\,d\ell'$$

$$\approx \sqrt{\frac{jk_o}{8\pi}}\int_C \frac{1}{\sqrt{R}}\,\hat{z}\cdot\hat{R}\times\vec{J}_s(\vec{r}')\,e^{-jk_o R}\,d\ell'$$

$$H_z^{scat\,ff}\left(\vec{r}\right)\approx\sqrt{\frac{jk_o}{8\pi}}\int_C\frac{1}{\sqrt{R}}\hat{z}\cdot\hat{R}\times\vec{J}_s\left(\vec{r}'\right)e^{-jk_oR}d\ell'$$



- In the far field $\left|\vec{r}\right|>>\left|\vec{r}'\right|$:
  - Magnitude approximation:
    - $\frac{1}{\sqrt{R}}\approx\frac{1}{\sqrt{r}}$ (where, $r=\left|\vec{r}\right|$)
  - Phase approximation:
    - $R\approx r-\hat{r}\cdot\vec{r}'$
    - $e^{-jk_oR}\approx e^{-jk_or}e^{jk_o\hat{r}\cdot\vec{r}'}$
  - Unit vector approximation:
    - $\hat{R}=\frac{\vec{R}}{R}\approx\hat{r}=\hat{x}\cos\phi+\hat{y}\sin\phi$
- Thus, in the far field:

$$H_z^{scat\,ff}\left(\vec{r}\right)\approx\sqrt{\frac{jk_o}{8\pi}}\frac{e^{-jk_or}}{\sqrt{r}}\hat{z}\cdot\hat{r}\times\int_C\vec{J}_s\left(\vec{r}'\right)e^{jk_o\hat{r}\cdot\vec{r}'}d\ell'$$

MoM Far Field Approximation

$$H_z^{scat\,ff}\left(\vec{r}\right)\approx\sqrt{\frac{jk_o}{8\pi}}\frac{e^{-jk_or}}{\sqrt{r}}\hat{z}\cdot\hat{r}\times\int_C\vec{J}_s\left(\vec{r}'\right)e^{jk_o\hat{r}\cdot\vec{r}'}d\ell'$$

- Expanding the current with the solution vector:

$$H_z^{scat\,ff}\left(\vec{r}\right)\approx\sqrt{\frac{jk_o}{8\pi}}\frac{e^{-jk_or}}{\sqrt{r}}\hat{z}\cdot\hat{r}\times\sum_{i=1}^{N}\alpha_i\int_{t_i}^{t_{i+1}}\hat{t}_ie^{jk_o\hat{r}\cdot\vec{r}'}d\ell'$$

- Using a mid-point rule to approximation the integration:

$$H_z^{scat\,ff}\left(r,\phi\right)\approx\sqrt{\frac{jk_o}{8\pi}}\frac{e^{-jk_or}}{\sqrt{r}}\hat{z}\cdot\sum_{i=1}^{N}\alpha_i\,\Delta t_i\left(\hat{r}\times\hat{t}_i\right)e^{jk_o\left(x_i^c\cos\phi+y_i^c\sin\phi\right)}$$

- where:

$$\hat{z}\cdot\left(\hat{r}\times\hat{t}_i\right)=\frac{\left(y_{i+1}-y_i\right)\cos\phi-\left(x_{i+1}-x_i\right)\sin\phi}{\Delta t_i}=\sin\left(\Omega_i\right)\cos\phi-\cos\left(\Omega_i\right)\sin\phi=\sin\left(\Omega_i-\phi\right)$$

  - where Ω_i is the angle the segment makes with the *x*-axis.

Echo Width

- Recall, the echo width of the target is:

$$\sigma_{2d}(\phi) = \lim_{k_o r\to\infty} 2\pi r \frac{P^s}{P^{inc}} = \lim_{k_o r\to\infty} 2\pi r \frac{\left|H_z^{scat}\right|^2 \eta_o}{\left|H_z^{inc}\right|^2 \eta_o}$$

- Given

$$H_z^{scat\,ff}(r,\phi) \approx \sqrt{\frac{jk_o}{8\pi}} \frac{e^{-jk_o r}}{\sqrt{r}} \sum_{i=1}^{N} \alpha_i \Delta t_i \frac{\left(\left(y_{i+1}-y_i\right)\cos\phi - \left(x_{i+1}-x_i\right)\sin\phi\right)}{\Delta t_i} e^{jk_o\left(x_i^c\cos\phi + y_i^c\sin\phi\right)}$$

- This leads to:

$$\boxed{\sigma_{2d}(\phi) = \frac{k_o}{4}\frac{1}{H_o^2}\left|\sum_{i=1}^{N}\alpha_i\left(\left(y_{i+1}-y_i\right)\cos\phi - \left(x_{i+1}-x_i\right)\sin\phi\right)e^{jk_o\left(x_i^c\cos\phi + y_i^c\sin\phi\right)}\right|^2}$$

  - where H_o is the amplitude of the incident field.