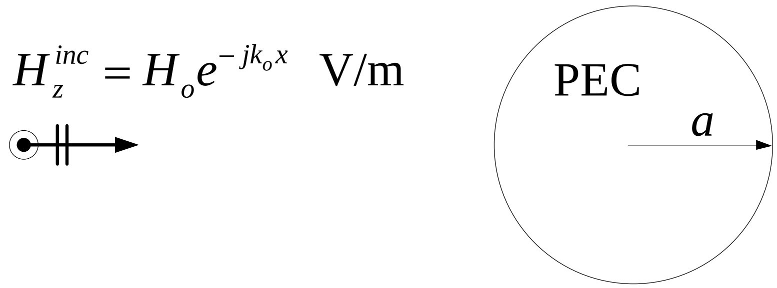
TE_z-Scattering by a Circular PEC Cylinder – Analytical Solution

$$\mu_o, \varepsilon_o$$

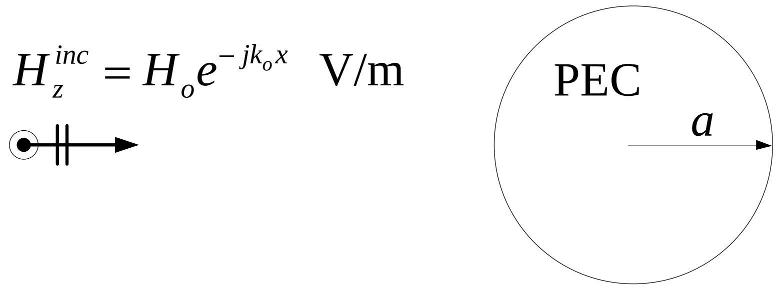
$$H_z^{inc} = H_o e^{-jk_o x} \quad \text{V/m}$$



- Consider a uniform plane wave traveling in the positive *x*-direction that is incident on a PEC cylinder of radius *a* situated in a homogeneous free space.
- Objective:
  - Compute the currents induced on the circular PEC cylinder
  - Compute the field scattered by the circular PEC Cylinder
- Analytical Solution
  - Based on an addition theorem expansion of the incident field into cylindrical harmonics
  - Posing the scattered field to be a superposition of cylindrical harmonics
  - Enforce the boundary condition on the surface of the cylinder
    $E_\phi^{tot}\left(\rho = a\right) = 0$

Cylindrical Harmonic Expansion of the Fields

μ_o, ε_o



- The incident field can be re-written via a cylindrical harmonic expansion:
  - $H_z^{inc} = H_o e^{-jk_o x} = H_o \sum_{n=-\infty}^{+\infty} j^{-n} J_n(k_o \rho) e^{jn\phi}$
- The incident electric field can then be expressed as:
  - $\vec{E}^{inc} = \dfrac{1}{j\omega\varepsilon_o} \nabla \times \vec{H}^{inc} = \dfrac{\eta_o}{jk_o}\left[\hat{\rho}\dfrac{1}{\rho}\dfrac{\partial H_z^{inc}}{\partial \phi} - \hat{\phi}\dfrac{\partial H_z^{inc}}{\partial \rho}\right]$
  - Expanding:
  - $E_\rho^{inc} = H_o \dfrac{\eta_o}{k_o \rho} \sum_{n=-\infty}^{+\infty} n j^{-n} J_n(k_o \rho) e^{jn\phi}, \quad E_\phi^{inc} = j\eta_o H_o \sum_{n=-\infty}^{+\infty} j^{-n} J_n'(k_o \rho) e^{jn\phi}$
    - where J_n' is the derivative with respect to the argument

Cylindrical Harmonic Expansion of the Scattered Fields

- In the region $\rho > a$, the scattered fields are also posed as cylindrical harmonic expansions:
    - $H_z^{scat} = H_o \sum_{n=-\infty}^{+\infty} c_n H_n^{(2)}(k_o\rho) e^{jn\phi}$
- The electric field is computed from the posed magnetic field:
    - $\vec{E}^{scat} = \frac{1}{j\omega\varepsilon_o} \nabla \times \vec{H}^{scat} = \frac{\eta_o}{jk_o}\left[\hat{\rho}\frac{1}{\rho}\frac{\partial H_z^{scat}}{\partial\phi} - \hat{\phi}\frac{\partial H_z^{scat}}{\partial\rho}\right]$
    - Expanding:
    - $E_\rho^{scat} = H_o \frac{\eta_o}{k_o\rho} \sum_{n=-\infty}^{+\infty} nc_n H_n^{(2)}(k_o\rho) e^{jn\phi}, \quad E_\phi^{scat} = j\eta_o H_o \sum_{n=-\infty}^{+\infty} c_n H_n^{(2)\prime}(k_o\rho) e^{jn\phi}$
- Our next task is to solve for the infinite set of unknowns c_n.

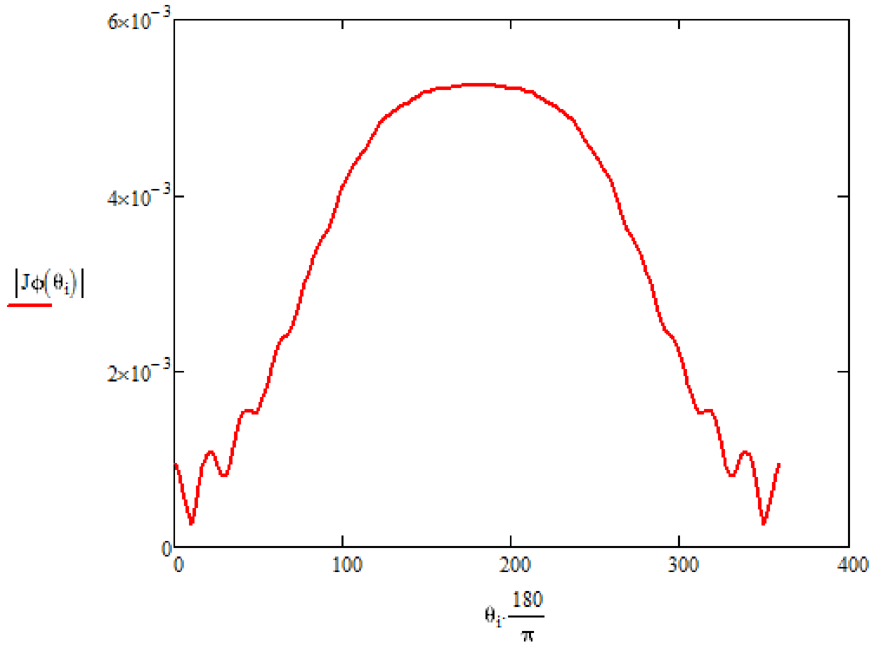
Boundary Condition

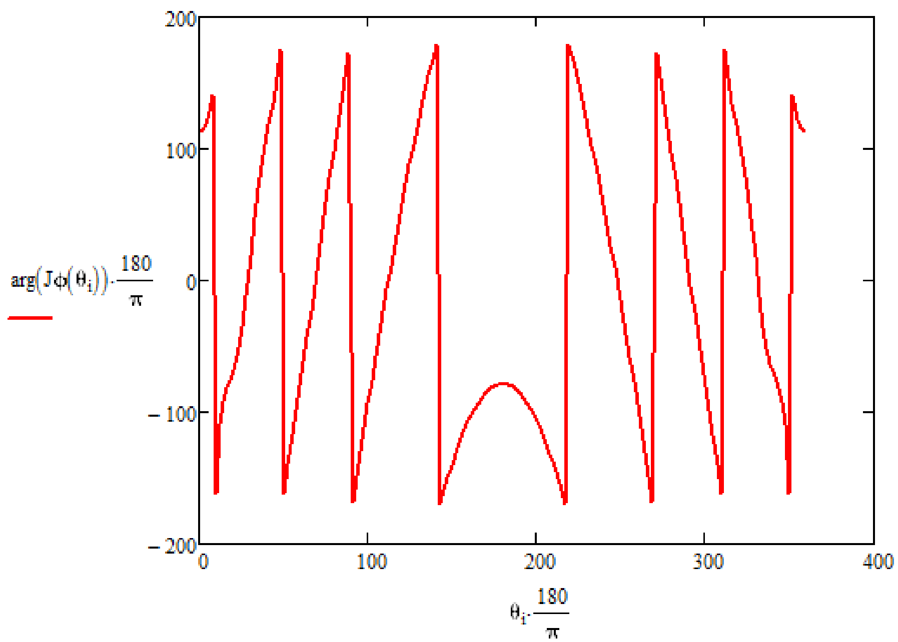
- Boundary condition:
  - $E_\phi^{tot}(\rho = a) = 0$
  - $E_\phi^{inc}(\rho = a) = -E_\phi^{scat}(\rho = a)$
- From the cylindrical harmonic expansion:
  - $j\eta_o H_o \sum_{n=-\infty}^{+\infty} j^{-n} J_n'(k_o a) e^{jn\phi} = -j\eta_o H_o \sum_{n=-\infty}^{+\infty} c_n H_n^{(2)\prime}(k_o a) e^{jn\phi}$
- Enforcing the orthogonality of the harmonics:
  - $c_n = -j^{-n} J_n'(k_o a) / H_n^{(2)\prime}(k_o a)$
- Therefore:
  - $$H_z^{scat} = -H_o \sum_{n=-\infty}^{+\infty} j^{-n} \frac{J_n'(k_o a)}{H_n^{(2)\prime}(k_o a)} H_n^{(2)}(k_o \rho) e^{jn\phi},$$
    $$E_\rho^{scat} = -H_o \frac{\eta_o}{k_o \rho} \sum_{n=-\infty}^{+\infty} n j^{-n} \frac{J_n'(k_o a)}{H_n^{(2)\prime}(k_o a)} H_n^{(2)}(k_o \rho) e^{jn\phi},$$
    $$E_\phi^{scat} = -j\eta_o H_o \sum_{n=-\infty}^{+\infty} j^{-n} \frac{J_n'(k_o a)}{H_n^{(2)\prime}(k_o a)} H_n^{(2)\prime}(k_o \rho) e^{jn\phi}$$

Solving for the Current Density

- The total magnetic field can be expressed as:
  - $H_z^{tot} = H_o \sum_{n=0}^{+\infty} j^{-n} \kappa_n \left(J_n(k_o\rho) - \frac{J_n'(k_o a)}{H_n'^{(2)}(k_o a)} H_n^{(2)}(k_o\rho) \right) \cos(n\phi), \quad \rho \ge a$
    - where, the infinite series have been modified to a semi-infinite series, &
      - $\kappa_n = \begin{cases} 1, & n = 0 \\ 2, & n > 0 \end{cases}$
- The surface current density induced on the PEC surface:
  - $\vec{J}_s = \hat{n} \times \left(\vec{H}^{tot} - 0 \right) \Big|_{\rho=a} = \hat{\rho} \times \hat{z} \vec{H}_z^{tot} \Big|_{\rho=a} = -\hat{\phi} H_z^{tot}(a,\phi)$
- Thus,
  - $\vec{J}_s = -\hat{\phi} H_o \sum_{n=0}^{+\infty} j^{-n} \kappa_n \left(J_n(k_o a) - \frac{J_n'(k_o a)}{H_n'^{(2)}(k_o a)} H_n^{(2)}(k_o a) \right) \cos(n\phi)$
- Recall the *Wronskian*:
  - $J_n'(k_o a) H_n^{(2)}(k_o a) - J_n(k_o a) H_n^{(2)\prime}(k_o a) = \frac{2j}{\pi k_o a}$

- The current density is then more conveniently written as:
  - $$\vec{J}_s = \hat{\phi}\frac{2jH_o}{\pi k_o a}\sum_{n=0}^{+\infty} j^{-n}\kappa_n \frac{\cos(n\phi)}{H_n'^{(2)}(k_o a)}$$
- This is the exact solution for the induced surface current density on a circular PEC cylinder excited by a TEz-polarized UPW traveling in the +*x* direction.
- Example: Circular PEC Cylinder, ka = 8.0





Computing the Far Field

- In the "far" scattered field, we assume
  - Region where the distance from the cylinder $(\rho - a)/\lambda >> 1$
  - More fundamentally: $k_o\rho >> 1$

- Asymptotic approximation of the Hankel function:
  - $\lim\limits_{k_o\rho\to\infty} H_n^{(2)}(k_o\rho) \approx \sqrt{\dfrac{2j}{\pi k_o\rho}}\, j^n e^{-jk_o\rho}$
- Therefore, in the far field:
  - $$\lim_{k_o\rho\to\infty} H_z^{scat}(\rho,\phi) = \lim_{k_o\rho\to\infty} -H_o \sum_{n=0}^{+\infty} j^{-n}\kappa_n \frac{J_n'(k_o a)}{H_n'^{(2)}(k_o a)} H_n^{(2)}(k_o\rho)\cos(n\phi)$$
    $$H_z^{scat-ff}(\rho,\phi) \approx -H_o\sqrt{\frac{2j}{\pi k_o\rho}}\, e^{-jk_o\rho}\sum_{n=0}^{+\infty}\kappa_n \frac{J_n'(k_o a)}{H_n'^{(2)}(k_o a)}\cos(n\phi)$$

Echo Width

- Recall, the echo width is defined as:
  - $$\sigma_{2D}(\phi) = \lim_{\rho\to\infty}(2\pi\rho\Delta z)\frac{\text{Scattered Power Density}}{\text{Incident Power Density}} = \lim_{\rho\to\infty} 2\pi\rho \frac{\left|H_z^{scat}\right|^2 \eta_o}{\left|H_z^{inc}\right|^2 \eta_o}$$
- In the far field:
  - $$\left|H_z^{scat-ff}(\rho,\phi)\right|^2 = H_o^2 \frac{2}{\pi k_o \rho}\left|\sum_{n=0}^{+\infty} \kappa_n \frac{J_n'(k_o a)}{H_n^{(2)\prime}(k_o a)}\cos(n\phi)\right|^2$$
- Therefore
  - $$\sigma_{2D}(\phi) = \lim_{\rho\to\infty} 2\pi\rho \frac{H_o^2 \frac{2}{\pi k_o \rho}\left|\sum_{n=0}^{+\infty} \kappa_n \frac{J_n'(k_o a)}{H_n^{(2)\prime}(k_o a)}\cos(n\phi)\right|^2 \eta_o}{H_o^2 \eta_o}$$
- or
  - $$\boxed{\sigma_{2D}(\phi) = \frac{4}{k_o}\left|\sum_{n=0}^{+\infty} \kappa_n \frac{J_n'(k_o a)}{H_n^{(2)\prime}(k_o a)}\cos(n\phi)\right|^2}$$

MathCad Example

$ka := 8 \qquad a := \frac{ka}{2\pi} \qquad ko := 2\cdot\pi$

$\eta := 376.7303134617706554679$

$Jnp(n,x) := Jn(n-1,x) - \frac{n}{x}\cdot Jn(n,x)$

$Hn2(n,x) := Jn(n,x) - j\cdot Yn(n,x)$

$Ynp(n,x) := Yn(n-1,x) - \frac{n}{x}\cdot Yn(n,x)$

$Hn2p(n,x) := Jnp(n,x) - j\cdot Ynp(n,x)$

$$\kappa(n) := \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{otherwise} \end{cases}$$

$$Nmax := \begin{cases} ceil(6\cdot ka) & \text{if } ceil[6(ka)] > 10 \\ 10 & \text{otherwise} \end{cases}$$

$$\sigma 2D(\phi) := \frac{4}{ko}\cdot\left[\left|\sum_{n=0}^{Nmax}\left(\kappa(n)\cdot\frac{Jnp(n,ka)\cdot\cos\left(n\cdot\phi\cdot\frac{\pi}{180}\right)}{Hn2p(n,ka)}\right)\right|\right]^2$$