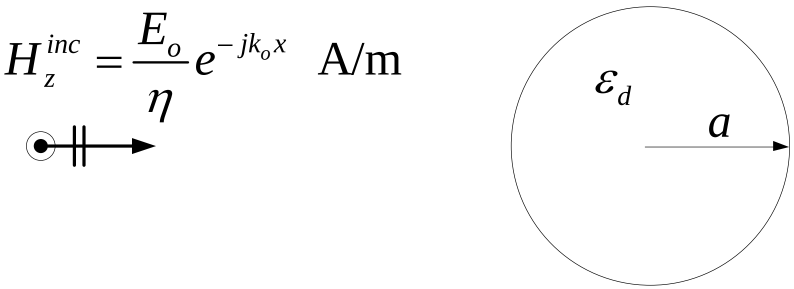
TE$_z$-Scattering by a Circular Dielectric Cylinder – Analytical Solution

μ_o, ε_o

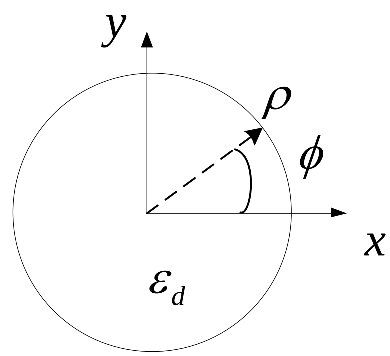
$$H_z^{inc} = \frac{E_o}{\eta} e^{-jk_o x} \text{ A/m}$$



- Consider a uniform plane wave traveling in the positive *x*-direction that is incident on a dielectric cylinder of radius *a* situated in a homogeneous free space.
- Objective:
  - o Compute the total fields within the circular dielectric cylinder
  - o Compute the total fields outside of the circular dielectric cylinder
- Analytical Solution
  - o Based on an addition theorem expansion of the incident field into cylindrical harmonics
  - o Posing the scattered field exterior to the cylinder as a superposition of cylindrical harmonics
  - o Posing the total field inside to be a superposition of cylindrical harmonics
  - o Enforce the boundary conditions on the surface of the cylinder

Cylindrical Harmonic Expansion of the Interior/Exterior Fields

$$H_z^{inc} = \frac{E_o}{\eta_o} e^{-jk_o x} \quad \text{A/m}$$



- The incident field:
  - $H_z^{inc} = H_o \sum_{n=0}^{+\infty} j^{-n} \kappa_n J_n\left(k_o\rho\right)\cos(n\phi), \qquad \kappa_n = 1(n=0), = 2(n \neq 0)$
  - $E_\rho^{inc} = \frac{1}{j\omega\varepsilon_o}\frac{1}{\rho}\frac{\partial H_z^{inc}}{\partial \phi} = -\frac{\eta_o H_o}{jk_o}\frac{1}{\rho}\sum_{n=0}^{+\infty} j^{-n} n\kappa_n J_n\left(k_o\rho\right)\sin(n\phi)$
  - $E_\phi^{inc} = \frac{-1}{j\omega\varepsilon_o}\frac{\partial H_z^{inc}}{\partial \rho} = j\eta_o H_o \sum_{n=0}^{+\infty} j^{-n}\kappa_n J_n'\left(k_o\rho\right)\cos(n\phi)$
- Procedure:
  - Pose H_z^{scat} $(\rho > a)$ and H_z^{tot} $(\rho < a)$ as cylindrical harmonic expansion
  - Compute $\vec{E}^{scat}$ $(\rho > a)$ and $\vec{E}^{tot}$ $(\rho < a)$ using Maxwell's equations
  - Solve for unknown coefficients by enforcing boundary conditions and orthogonality of the harmonics.

Cylindrical Harmonic Expansion of the Scat/Tot Fields

- Scattered field ($\rho > a$):
  - $H_z^{scat} = H_o \sum_{n=0}^{+\infty} a_n j^{-n} \kappa_n H_n^{(2)}\left(k_o \rho\right) \cos(n\phi),\ \rho > a$
  - $E_\phi^{scat} = j\eta_o H_o \sum_{n=0}^{+\infty} a_n j^{-n} \kappa_n H_n^{(2)\prime}\left(k_o \rho\right) \cos(n\phi)$
- Total field ($\rho < a$):
  - $H_z^{tot} = H_o \sum_{n=0}^{+\infty} b_n j^{-n} \kappa_n J_n\left(k_d \rho\right) \cos(n\phi),\ \rho < a$
  - $E_\phi^{tot} = j\eta_d H_o \sum_{n=0}^{+\infty} b_n j^{-n} \kappa_n J_n'\left(k_d \rho\right) \cos(n\phi)$
    - where $k_d = \omega\sqrt{\varepsilon_d \mu_o}$ and $\eta_d = \sqrt{\dfrac{\mu_o}{\varepsilon_d}}$
- a_n and b_n are unkowns.
  - How do we solve for them?

Solving for the Unknown Coefficients

- Boundary conditions:
  - $H_z^{inc}\left(\rho=a^+\right)+H_z^{scat}\left(\rho=a^+\right)=H_z^{tot}\left(\rho=a^-\right)$
  - $E_\phi^{inc}\left(\rho=a^+\right)+E_\phi^{scat}\left(\rho=a^+\right)=E_\phi^{tot}\left(\rho=a^-\right)$
- Applying orthogonality, this leads to:

$$H_o j^{-n}\kappa_n J_n\left(k_o a\right)\cos(n\phi)+H_o a_n j^{-n}\kappa_n H_n^{(2)}\left(k_o a\right)\cos(n\phi)=H_o b_n j^{-n}\kappa_n J_n\left(k_d a\right)\cos(n\phi)$$

$$j\eta_o H_o j^{-n}\kappa_n J_n'\left(k_o a\right)\cos(n\phi)+j\eta_o H_o a_n j^{-n}\kappa_n H_n^{(2)\prime}\left(k_o a\right)\cos(n\phi)=j\eta_d H_o b_n j^{-n}\kappa_n J_n'\left(k_d a\right)\cos(n\phi)$$

- Canceling common terms, and re-writing as a matrix equation:
  - $$\begin{bmatrix} H_n^{(2)}\left(k_o a\right) & -J_n\left(k_d a\right) \\ \eta_o H_n^{(2)}\left(k_o a\right) & -\eta_d J_n'\left(k_d a\right)\end{bmatrix}\begin{bmatrix} a_n \\ b_n\end{bmatrix}=\begin{bmatrix} -J_n\left(k_o a\right) \\ -\eta_o J_n'\left(k_o a\right)\end{bmatrix}$$
- The 2×2 can be solved for a_n and b_n:
  - $$a_n=\frac{\eta_d J_n'\left(k_d a\right)J_n\left(k_o a\right)-\eta_o J_n'\left(k_o a\right)J_n\left(k_d a\right)}{\eta_o J_n\left(k_d a\right)H_n^{(2)\prime}\left(k_o a\right)-\eta_d J_n'\left(k_d a\right)H_n^{(2)}\left(k_o a\right)}$$
  - $$b_n=\eta_o\frac{J_n\left(k_o a\right)H_n^{(2)\prime}\left(k_o a\right)-J_n'\left(k_o a\right)H_n^{(2)}\left(k_o a\right)}{\eta_o J_n\left(k_d a\right)H_n^{(2)\prime}\left(k_o a\right)-\eta_d J_n'\left(k_d a\right)H_n^{(2)}\left(k_o a\right)}$$

Computing the Near-Field

- The total magnetic field H_z^{tot} can be written as:
  - $$H_z^{tot} = \begin{cases} H_o \sum_{n=0}^{+\infty} b_n j^{-n} \kappa_n J_n\left(k_d \rho\right) \cos(n\phi), & \rho \le a \\ H_o \sum_{n=0}^{+\infty} j^{-n} \kappa_n \left[J_n\left(k_o \rho\right) + a_n H_n^{(2)}\left(k_o \rho\right) \right] \cos(n\phi), & \rho \ge a \end{cases}$$
- To validate the MoM solution inside the cylinder, $\vec{E}^{tot}$ is needed
  - $$E_\rho^{tot}\left(\rho,\phi\right) = \frac{1}{j\omega\varepsilon_d} \frac{1}{\rho} \frac{\partial H_z^{tot}}{\partial \phi} = -\frac{\eta_d H_o}{jk_d} \frac{1}{\rho} \sum_{n=0}^{+\infty} j^{-n} n b_n \kappa_n J_n\left(k_d \rho\right) \sin(n\phi)$$
  - $$E_\phi^{inc} = \frac{-1}{j\omega\varepsilon_d} \frac{\partial H_z^{inc}}{\partial \rho} = j\eta_d H_o \sum_{n=0}^{+\infty} j^{-n} b_n \kappa_n J_n'\left(k_d \rho\right) \cos(n\phi)$$
- Using Cartesian projections:
  - $$E_x^{tot}\left(\rho,\phi\right) = \cos\phi E_\rho^{tot}\left(\rho,\phi\right) - \sin\phi E_\phi^{tot}\left(\rho,\phi\right)$$
  - $$E_y^{tot}\left(\rho,\phi\right) = \sin\phi E_\rho^{tot}\left(\rho,\phi\right) + \cos\phi E_\phi^{tot}\left(\rho,\phi\right)$$

Computing the Far Field

- Often it is desirable to compute the "far" scattered field.
  - Region where the distance from the cylinder $(\rho - a)/\lambda >> 1$
  - Also quantified as the region where $k_o\rho >> 1$

- Asymptotic approximation of the Hankel function:
  - $\lim\limits_{k_o\rho\to\infty} H_n^{(2)}(k_o\rho) \approx \sqrt{\dfrac{2j}{\pi k_o\rho}}\, j^n e^{-jk_o\rho}$
- Therefore, in the far field:
  - $$\lim_{k_o\rho\to\infty} H_z^{scat}(\rho,\phi) = \lim_{k_o\rho\to\infty} H_o \sum_{n=0}^{+\infty} j^n (-1)^n \kappa_n a_n H_n^{(2)}(k_o\rho)\cos(n\phi)$$
    $$\approx H_o \sqrt{\frac{2j}{\pi k_o\rho}} e^{-jk_o\rho} \sum_{n=0}^{+\infty} j^{2n} (-1)^n \kappa_n a_n \cos(n\phi)$$
- Therefore,
  - $H_z^{scat-ff}(\rho,\phi) = H_o \sqrt{\dfrac{2j}{\pi k_o\rho}} e^{-jk_o\rho} \sum\limits_{n=0}^{+\infty} \kappa_n a_n \cos(n\phi)$

Echo Width

- The echo width is the equivalent area proportional to the apparent size of the target that scatters the field in a given direction relative to the power incident from another direction.
  - $$\sigma_{2D}(\phi) = \lim_{\rho\to\infty}(2\pi\rho\Delta z)\frac{\text{Scattered Power Density}}{\text{Incident Power Density}} = \lim_{\rho\to\infty} 2\pi\rho \frac{\left|H_z^{scat}\right|^2 \eta_o}{\left|H_z^{inc}\right|^2 \eta_o}$$
- In the far field:
  - $$\left|H_z^{scat-ff}(\rho,\phi)\right|^2 = H_o^2 \frac{2}{\pi k_o \rho}\left|\sum_{n=0}^{+\infty}\kappa_n a_n \cos(n\phi)\right|^2$$
- Therefore
  - $$\sigma_{2D}(\phi) = \lim_{\rho\to\infty} 2\pi\rho \frac{H_o^2 \frac{2}{\pi k_o \rho}\left|\sum_{n=0}^{+\infty}\kappa_n a_n \cos(n\phi)\right|^2 \eta_o}{H_o^2 \eta_o}$$
- or
  - $$\boxed{\sigma_{2D}(\phi) = \frac{4}{k_o}\left|\sum_{n=0}^{+\infty}\kappa_n a_n \cos(n\phi)\right|^2}$$

MathCad Example

$ka = 4 \qquad \eta = 376.73031346177066 \qquad \varepsilon r = 1.2 \qquad \eta d := \frac{\eta}{\sqrt{\varepsilon r}}$

$$an(n) := \frac{\eta d \cdot Jnp(n,kda) \cdot Jn(n,ka) - \eta \cdot Jnp(n,ka) \cdot Jn(n,kda)}{\eta \cdot Jn(n,kda) \cdot Hn2p(n,ka) - \eta d \cdot Jnp(n,kda) \cdot Hn2(n,ka)}$$

$$\sigma 2dEx(\phi) := 10\log\left[\frac{4}{ko} \cdot \left[\left|\sum_{n=0}^{Nmax}\left(\kappa(n) \cdot an(n) \cdot \cos\left(n \cdot \phi \cdot \frac{\pi}{180}\right)\right)\right|\right]^2\right]$$